MATHEMATICS IN ANCIENT AND MEDIEVAL INDIA

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MATHEMATICS IN ANCIENT AND MEDIEVAL INDIA

Dr. A. K. BAG, M. Sc., Ph. D.
Project Coordinator (History of Science)
Indian National Science Academy,
Bahadur Shah Zafar Marg
New Delhi-110002

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INTRODUCTION

The present work deals with an account of the development of mathematics in Ancient and Medieval India. It is divided into seven chapters relating respectively to Scholars of Mathematics, Arithmetic, Geometry, Algebra, Trigonometry, Infinitesimal Calculus and Trend of Indo-Persian/Arabic literature on Mathematics in the Medieval Period. In these chapters an account of the Indian works in mathematics and a critical review of the subject as well as some mathematical techniques, concepts and methods in ancient and medieval India, which have not hitherto received attention from modern scholars have been discussed and interpreted. Comparative developments in other countries and their chronological relationship have also been referred to wherever necessary.

Expression of very large numbers by means of indices to ten like that of Indian workers, and their use of fractions of various types were not accessible in the work of any contemporary nations. The concept of nine numerals and the decimal place-value with the introduction of zero is a significant Indian contribution to the development of Arithemetic. A systematic and straight-forward technique of fundamental mathematical operations and formulation of rule of three methods of calculation have been found to be of Indian origin. Reference has also been made to the comparatively more accurate results given by the Indian workers for the surd numbers. The geometrical knowledge of the ancient Indians have been derived from the instruments, units and the shape of the bricks used in altar constructions. Technical terms as found in the *Śulba-sūtras* and in later works are coined which interpret basic concepts of Indian nomenclature. A clear general statement of Pythagoras' Theorem, calculation of the value of $\sqrt{2}$ by geometrical method and the classification of geometrical figures on the basis of angles and sides have been pointed out to be notable contribution to geometry by the Indians in the age of *Śulba-sūtras* (600 B. C.). Brahmagupta's theorem relating to the determination of diagonals of cyclic quadrilateral and his methods for the construction of a quadrilateral may also be regarded as remarkable achievements of Indian geometry. The growth of algebra in India has been discussed with particular reference to symbols of operation, equation, rule of false position, quadratic equation, indeterminate equations of first and second degree, progressive series, permutation and combination and Pascal's triangle with binomial theorem. A comprehensive interpretation of *meru-prastāra* scheme (10th century A. D.) due to Halāyudha (commentator of Piṅgala's *Chandaḥ Sūtra* —200 B. C.) as forming the basis of binomial expansion for any positive integral exponent in algebra has been given. The development of trigonometrical knowledge in ancient and medieval India has been discussed with particular reference to trigonometrical formulae, sine table, value of π and trigonometrical series. It has been shown that the Indians made a notable addition to the knowledge of trigonometry after Ptolemy relating to trigonometrical formulae which was systematised into a special branch of knowledge named *jyotpatti* (trigonometry). The work on trigonometrical series—π, sine, cosine and tan was first carried out by the Indians more than a century before Newton (1664 A. D.) and Leibniz (1676 A.D.). The Indian concepts of infinitesimal calculus both differential and integral appeared in the work of Bhāskara II (1150 A. D.). These are similar to those of Archimedes (c. 225 B. C.), and shows in some respects an improvement upon Archimedes' as to the infinitesimal character of the units chosen. Further development of this concept of infinitesimal (integral) calculus occurred in *Yuktibhāṣā* in connection with the summation of infinite series (π—series) more than a century before Newton and Leibniz. The trend of Indo-Persian/Arabic literatures in the Medieval period in India has been discussed in a nut shell. The work closes with a general bibliography which serves as an important source in addition to original manuscript sources.

I take this opportunity to express my gratitude to Prof. P. Ray for his varied help and constant inspiration in the study of history of mathematics. My grateful thanks are due to Prof. J. N. Mukherjee, Sri S. N. Sen, Prof. M. C. Chaki without whose encouragement, I would not venture to complete this work. I am also grateful to Prof. and Mrs. J. L. Bhaduri for their kind interest in the work. Thanks are also due to the National Library (Calcutta), Central Library of the Calcutta University, and specially to the employees of the Asiatic Society for giving me spontaneous help and ungrudging cooperation in using their library for my work. To conclude, I also like to place on record that the printing of the work was started in 1971 by C. S. S. but was stopped for the last several years until it was taken up by one of its partners, Chaukhambha Orientalia. However, the new concern has taken enough care to get it out in quickest possible time. My thanks are also due to them.

June 27, 1979 **A. K. BAG**

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MATHEMATICS IN ANCIENT
AND
MEDIEVAL INDIA

CHAPTER 1

SCHOLARS OF MATHEMATICS

India had a glorious past in the field of mathematics. There is no doubt that there was a time when the great scholars had been adored and those desirous of learning would flock around them for knowledge. Hence it is of interest to know the names of these scholars and their activities who are the pioneers in this branch of science. For systematic study of the growth of mathematics in ancient and medieval India, it will be convenient to pursue the development in successive stages like, 1. Vedic Period (c. 1500 B.C. to 200 B.C.), 2. Post-Vedic Period (c. 200 B.C. to 400 A.D.), 3. Early Medieval Period (c. 400 A.D. to 1200 A.D.) and 4. Late Medieval Period (c. 1200 A.D. to 1800 A.D.).

1. VEDIC PERIOD—Origin of Mathematical Knowledge (c. 1500 B.C. to 200 B.C.)

About two thousand years before the Christian era, the Indus Valley was invaded by an Aryan race. Following this, at about 1500 B.C., a crude civilisation known as the Vedic civilisation began to emerge in India.

The Vedic civilisation has four stages of development viz., (1) the *Saṃhitā* (the four *Saṃhitās*, namely *Ṛk*, *Sāma*, *Yajus* and *Atharvan*), (2) the next stage of the *Brāhmaṇa* (theological and ritual treatises), (3) then the stage of the *Āraṇyakas* (metaphysical appendices of the *Brāhmaṇas*) and *Upaniṣads* (philosophical texts regarding the absolute one), and

(4) the last stage of the *Vedāṅgas*. Information related to mathematical ideas in the literatures of the first three stages of the Vedic age is meagre. The *Vedāṅga* literatures are of six types namely : (i) *śikhṣā* (the science of the pronunciation of letters, accents ......), (ii) *kalpa* (containing rules for ceremonial and sacrificial ceremony), (iii) *vyākaraṇa* (grammar), (iv) *nirukta* (etymology), (v) *chanda* (the science of metres) and (vi) *jyautiṣa* (astronomy). The literatures of the *Vedāṅgas* are known as a whole Sūtra literatures, because they are written in forms of short rule, a peculiar style of composition characterised by utmost brevity and rigid systematization to preserve the cultural heritages of the Brāhmaṇas in a manageable form.

Sūtra Period, an age of specialisation—It may be noted in this connection that the study of mathematics started with the sūtra period. At first, the study was strictly subservient to the primary needs, and education meant only the transmission of traditions from the teacher to the pupil and the committing to memory the sacred texts. In course of time, however, the contents of this education began to widen out and each one of the several *aṅgas* of the Veda began to develop. It is in this connection with the construction of sacrificial altars of proper size and shape that the problems of geometry and perhaps also of arithmetic and algebra were evolved. The study of astronomy arose out of the necessity for fixing the proper time for sacrifices.

Vedic Sources on Mathematics—The critical mathematical knowledge of Sūtra literature led us to assume that there were mathematical works of even earlier age but they are lost. The names of seven śulbakāras are known. They are : Baudhāyana, Āpastamba, Kātyāyana, Mānava, Maitrāyana, Vārāha and Hiraṇyakeśī. The śulba works by these authors give methods and solutions of various problems of construction of altars required for Vedic sacrifices. The authors may be regarded as geometrician of the Vedic times. The *śulbasūtras* (connected to Śrauta section of the Kalpasūtra) of Baudhāyana, Āpastamba, Kātyāyana, Mānava, Maitrāyana, Vārāha and Hiraṇyakeśī are the main sources of our Vedic mathematics[1].

There is some controversy about the birth place of Baudhāyana, the oldest *śulbakār*, some locating it in the Andhra country in the South while others in the Āryāvarta, the region between the river Gaṅgā and Yamunā[2]. His *Baudhāyanaśulbasūtra*[3] is complete in three chapters. Baudhāyana never claimed that he was the first to discover the principles of geometry or applied them to the problems of construction of altars. He gave a general enunciation of the Pythagoras' theorem, an approximate value of $\sqrt{2}$ correct to five places of decimal, construction of a square equal to sum or difference of two squares, various methods of transformation of one figure to another etc. The native place of next *śulbakār* Āpastamba has been located similarly in a controversial manner either in

1. Beside these śulbas, we get the names of two other śulbas, namely *Vādhula* and *Māsaka*. Vide. B.Datta. *The Science of Śulba*, p.2, Calcutta University, 1932.
2. Ramgopal. *India of Vedic Kalpasūtras*. National Publishing House, Delhi, p. 100,1959.
3. Ed. with English translation and notes by G. Thibaut with the commentary of Dvārakānātha Yajvā, *Pandit*, Old Series, *9* and *10*, 1874-75; n.s. *1*, Benaras, 1877; Ed. by W. Caland, Vol.3, Calcutta, 1913.

the Andhra country or in Kurupāncāla[1]. His *Āpastambaśulbasūtra*[2] is complete in six chapters. It gives in more detail the similar constructions, geometrical propositions and rules applicable for other constructions. The *śulbasūtras* of Kātyāyana[3] and Mānava[4] having similar contents are published, and the remaining śulbas are still in manuscript form. The *śulbasūtras* of Baudhāyana, Āpastamba and Kātyāyana belong to the former class and these give us a picture of development of early geometry and mathematics in India before the rise and advent of Jaina sect (c. 500-300 B.C.). The śulbasūtras of Mānava, Maitrāyana, Vārāha and Hiraṇyakeśī which belong to the latter class add very little to the knowledge of information available in this respect.

From the view point of mathematical contribution, the *Vedāṅga jyautiṣa*[5], one of the Vedāṅgas deal-

1. Ramgopal, *ibid.*, p.96.
2. Ed. with German translation and notes by A. Burk. *Zeitschrift d. Deutsch. Morgenlandischen Gesellschaften*, *55*, p.543-91, 1901; *56*, p. 327-91, 1902; Ed. by D. Śrinivāsan Char and V. S.Narasimhachar, Mysore Sanskrit Series no. 73, 1931, etc.
3. Ed. with English translation (a portion only) by G. Thibaut, *Pandit*, n.s. *4*, 1882; Ed. with the commentary of Karkācārya, vide *Kātyāyana Śrautasūtra*, Ed. by M.M.Pāṭhaka, Benaras, 2 Vols. 1900; Published in the Kāśī Sanskrit Series no. 120 with Karka's *Bhāṣya* and Mahīdhara's commentary, Benaras, 1936, etc.
4. Ed. with English translation by J.M.Van Gelder, New Delhi; (Śatapiṭaka series no. 17), 1961.
5. Ed. by A. Weber, Berlin, 1862; Ed. (a portion) by G. Thibaut with notes, *JASB*, *46*, 1877; Ed. by Sudhākara Dvivedi with *Bhāṣya* of Somākara Śeṣa and S. Dvivedi and Muralidhara's explanatory notes, Bombay, 1908; Ed. with English translation by R. Shyama Sastri, Mysore, 1936; Ed. by A. Weber with a preface by Jagannath Tripathi, Allahabad, 1960 etc.

ing with astronomy, in its three recensions namely, *Ārca jyautiṣa*, *Yājuṣa jyautiṣa* and *Atharva jyautiṣa* may also be regarded as one of the sources of Vedic mathematics. A great deal of controversy prevailed about the time of the *Vedāṅga jyautiṣa*, but the modern scholars are more or less unanimous and fix it in 200 B.C.

2. POST-VEDIC PERIOD—Period of Arrested Progress and Some Foreign Contacts (c. 200 B.C. to 400 A.D.)

During this period, no original contribution to the development of astronomy and mathematics in the strictest sense was made. But several works on astronomy and mathematics were composed on the basis of knowledge of the earlier Vedic period and there is also evidence of foreign contacts.

(a) *Jaina works* : Amongst the religious works of the Jainas, those that are important from the view point of mathematics are : *Sūryaprajñapti*,[1], *Sthānāṅgasūtra*, *Bhagavatīsūtra*,[2], *Tattvārthādhigamasūtra*[3] of Umāsvātī, *Anuyogadvārasūtra*[4], *Kṣetrasamāsa*, *Trilokasāra*[5], etc. There were certainly other mathematical

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1. Ed. with commentary of Malaya Giri. Agamadaya Samiti, 1918; Ed. by Amolakrisi with Hindi translation, Hyderabad; Ed. with English translation and notes by G. Thibaut, *Journal of the Asiatic Society of Bengal (JASB), 49*, p. 107-27, p. 181-206, 1880.
2. Ed. by N. L. Vaidya with the commentary of Abhayadeva Sūri, *Śrīvijayadeva Sursaṅgha* Series no. 8, 1954.
3. Ed. by K.P.Mody, Calcutta, 1903; By H.R.Kapadia, Bombay, 1926; By A. S. Sastri with the commentary *Sukhabodha* of Bhāskaranandi, *Govt. Oriental Library Sans. Series* No. 84, Mysore, 1944.
4. Ed. with Hemacandra's commentary, *Jaina Pustakoddhāra Series* no.37, Bombay, 1916.
5. Ed. by Nathurama Premi with the commentary of Mādhava Candra, Bombay, 1919.

treatises by the early Jaina scholars, which are now lost. Considerable amount of search and research about the ancient works of the Jainas is necessary. We get the names of Siddhasena, Bhadrabāhu and many others who quoted mathematical formulae in connection with their doctrines but were not mathematicians by themselves.

The literature of the Jainas is generally classified into four groups[1] viz. (i) *Dharmakathanuyoga* (or the exposition of the principle of religion), (ii) *Gaṇitānuyoga* (or exposition of the principle of mathematics), (iii) *Saṃkhyāna* (or science of numbers) and (iv) *Jyotiṣa* (or astronomy). According to them, a child should be taught firstly writing, then arithmetic. The *arithmetic* and *jyotiṣa* had been considered as one of the main accomplishment of a Jaina saint.

Kusumpura school[2] : Umāsvātī (150 B.C.), one of the greatest Jaina metaphysician of India first cites the existence of this school of mathematics. He resided in the city of Kusumpura (ancient Pataliputra) near Patna. Probably the school was originated long before the time of famous Jaina saint Bhadrabāhu (300 B.C.) who lived at Kusumpura. The culture of mathematics and astronomy survived in this school for many centuries, where Āryabhaṭa I perhaps took his lesson in the fifth century A.D.

(b) *Hindu works* : Eighteen siddhāntas[3] were

1. Kapadia, H. L. Preface of *Gaṇitatilaka*, Ed. by H. R. Kapadia Gaekward Oriental Series no. 78, Baroda, 1937.
2. Datta, B., "The Jaina School of Mathematics", *Bulletin of the Calcutta Mathematical Society* (*BCMS*), *21*, p. 126-8, 1929.
3. *Sūryasiddhānta*, Ed. by Burgess, vide. Introduction of P. C. Sengupta, p. VIII, fn. University of Calcutta, 1935.

composed during this period. They go after the name of their authors namely Sūrya, Paitāmaha, Vyāsa, Vaśiṣṭha, Atri, Parāśara, Kāśyapa, Nārada, Garga, Marīci, Manu, Aṅgira, Lomaśa, Pauliśa, Cavana, Yavana, Bhṛgu and Śaunaka. Only five siddhāntas namely, the *Pauliśa Siddhānta*, *Romaka Siddhānta* (*Yavana Siddhānta*), *Vaśiṣṭha Siddhānta*, *Saurya Siddhānta* (i.e. *Sūrya Siddhānta*) and the *Paitāmaha Siddhānta* and a few others have survived, and the remainings are lost. These works also have gone through different recensions. It is now very difficult to assert what original matters are retained in their present redactions.

The modern *Sūrya Siddhānta*[1] which is a standard astronomical treatise is of particular interest from mathematical point of view. It is divided into 14 chapters and its astronomical calculations are accepted and followed throughout India. The original *Sūrysiddhānta* which is now lost, began to take shape from about the fourth or fifth century A.D. at a time when old astronomical ideas and calculations came to be revised and placed on a scientific and mathematical basis. A rough idea of the contents of this work may be had from chs. i, ix, x, xi, xvi and xvii of the *Pañcasiddhāntikā* of Varāhamihira. The text in its present form is of much later development and result of many corrections and interpolations. There is a considerable agreement between the *Sūryasiddhān-*

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sūrya pitāmaho vyāso vaśiṣṭho'triḥ parāśaraḥ
kāśyapo nārado gargo marīcirmanuraṅgiraḥ ||
lomaśaḥ pauliśaścaiva cyavano yavano bhṛguḥ
śaunako'ṣṭādaśascaite jyotiḥśāstrapravartakāḥ ||

1. Translated with notes into English by E. Burgess with introduction by P. C. Sengupta, University of Calcutta, 1935.

ta as described by Varāhamihira and the modern *Sūryasiddhānta.* P.C. Sengupta has shown, the old version was amended by Varāhamihira himself in accordance with the teaching of Āryabhaṭa I (vide Āryabhaṭa's lost work, *BCMS*, *22*, 1930; Introduction to Burgess' *Sūryasiddhānta*, Calcutta Univ. 1935). The *Sūryasiddhānta* gives in addition various information on trigonometry.

3. EARLY MEDIEVAL PERIOD—Period of Revival and Maximum Activity of Indian Mathematics with Systematisation (c. 400-1200 A.D.)

Several modern historians of mathematics namely, Kaye[1], Cajori[2], Smith[3], etc. are almost unanimous in presuming that the revival in Hindu mathematics is due to the Greek contact. From a thorough discussion of all those matters, Burgess[4], concluded that the transmitting knowledge was pre-Ptolemian Greek astronomy of Hipparchus and it took place in the 5th and 6th centuries of the Christian era. Biot followed by Thibaut[5] is of opinion that the Greek astronomy was not transmitted to India through scientific treatises but through manuals used by Greek astrologers and calendar makers. P.C. Sengupta has thoroughly re-examined the whole subject and rejected the hypothesis that Indian astronomy was derived from the Greek sources. He says, "if Indian astronomy is to be held indebted to any foreign system of astronomy, that system of astronomy was the Babylonian astronomy—the fountain head from which

1. Kaye, G. R. *Indian Mathematics*, Calcutta, p. 9, 1915.
2. Cajori, F. *Hist. of Math.*, p.84, also p.83. Revised ed.
3. Smith, D. E. *Hist. of Math.*, *1*, p. 145, 2nd ed., 1923.
4. *Journal of the American Oriental Society*, *6*, p. 471, 1857.
5. Thibaut and Dvivedi, *Pañcasiddhāntikā*, Introduction p. 19.

both the systems took their rise"[1]. The specific instances of the influence of the Babylonian astronomy on the Indian have been given by Neugebauer[2] and Sen[3]. My intention in referring to this complex situation is not to dispute the hypothesis of the Greek contact with Hindu astronomy. Indeed, there is incontrovertible evidence of foreign contact. In the opinion of Datta[4] however the renaissance in the Hindu astronomy is entirely indigenous.

Whatever be the causes of the revival, their effect on the scope of the science of the *gaṇita* in this period was great. Astronomy, which had been a branch of Mathematics, separated out and began to be regarded by the name *jyotiṣa*. Geometry which belonged to a separate group of sciences viz. the *Kalpasūtra*, came to be regarded as an integral part of the *gaṇita*. Thus the readjusted science of the *gaṇita* consisted mainly of arithmetic, algebra and geometry[5]. Trigonometry was probably discovered by that time and came within the scope of *jyotiṣa*. Amongst the Jainas and the Buddhists the *gaṇita* was known by the name *saṃkhyāna* (or science of numbers).

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1. Sengupta, P. C. "Āryabhaṭa–the father of Indian Epicyclic Astronomy" *Journal of the Department of Letters* (*JDL/CU*), *18*, p. 55, 1929; vide also S. K. Das "Alleged Greek Influence on Hindu Astronomy; *Indian Historical Quarterly (IHQ)*, p. 68-77, 1928.
2. Neugebauer, O. *The Exact Sciences in Antiquity*, Princeton, 1952.
3. Sen, S. N. "Transmission of Scientific ideas............"*Bulletin of the National Institute of Sciences of India*, *21*, p. 13, 1963.
4. Datta, B. "The Scope and Development of Hindu Gaṇita", *Indian Historical Quarterly (IHQ)*, *5*, p. 486, foot note, 1929.
5. Vide, the works of Āryabhaṭa I, Brahmagupta, Bhāskara II and others.

Schools of Ujjain and Mysore : In this period, the mathematical activity centered mainly round three schools namely, Kusumpura, Ujjain and Mysore. Āryabhaṭa I took his lesson in the Kusumpura school Varāhamihira, Brahmagupta and Bhāskara II (Bhāskarācārya) belonged to the Ujjain school while Mahāvīra may be said to represent the Southern school of Mysore.

The relation of the two schools of Ujjain and and Mysore with Kusumpura is very interesting. A terrible famine is said to have raged for twelve years and devasted completely the Magadha country. As a result, one section of the Jaina community of Magadha headed by their priest Bhadrabāhu emigrated to South India (Mysore). On his way he passed through Ujjain and halted there for some time. This story is supported by the local tradition, several inscriptions (of 650 A.D.) and literature.[1]. The connection between these three important schools of Hindu mathematics might thus be said to have been established in a very early times.

Some scholars[2] have claimed Āryabhaṭa I and Bhāskara I as residents of Kerala on the grounds that practically all the important astronomical and mathematical works produced in Kerala follow the Āryabhaṭa school, and works of these ancient authorities have been very popular in that part of India as indicated by the existence of a large number of their manuscripts and commentaries there on. Bhāskara I's reference to *Āryasiddhānta* as *Asmaka-*

1. *BCMS*, *21*, p.128, 1929.
2. Shukla, K. S. English translation of the *Mahābhāskarīya*, p.2, Lucknow Univ., 1960; Srinivasiengar, C. N. *The History of Ancient Indian Mathematics*, p. 41, World Press, Calcutta, 1967.

tantra also testifies that the work was written in Asmakadeśa in Kerala. But from Āryabhaṭa I's own statement, *kusumpure abhyarcitam jñānam (Āryabhaṭīya*, Gaṇita, v. 1 *)* it is not unlikely to presume that after his lesson was over at the school of Kusumpura, Āryabhaṭa returned to his native place at Kerala where he established a school and wrote his mathematical works. Bhāskara I was more popular in the South India than in the North (vide Bhāskara I) but there is no mention of any place in Kerala as the seat of his activities.

BAKSHĀLĪ MANUSCRIPT (c. 400 A.D.)

This manuscript[1] written in Sāradā script was first discovered, in course of excavation by a farmer at Bakshālī, a village near Peshwar in 1881 A.D. It attracted the attention of H. Hoernlé[2], G.R.Kaye[3] and Datta[4], who examined the manuscript from various aspects but failed to come to a satisfactory conclusion in ascertaining its age. Hoernlé considered it as 3rd to 4th century A.D. manuscript, which has been accepted as fair by Bühler, Cantor and Cajori. But Kaye placed it in the 12th century A.D. Datta opines that since the direct evidence is absent, there will be no better evidence than to depend on historical grounds. In this respect, the mathematical principles, symbols and terminology employed in the work would be better guides. Accor-

1. Edited with Introduction by G. R. Kaye, Part I & II, New Imperical Series no. 43, Calcutta, 1927.
2. 'Notices under History of Asiatic Society for 1882', *Indian Antiquary, 12*, p. 89-90, 1883; 'The Bakshali Ms', *Indian Antiquary, 17*, p. 33-48, p. 275-79, 1888.
3. p. 74-77 (vide his edition); *JASB, 8*, p. 349-61, 1912.
4. Datta, B. "The Bakshali Mathematics", *Bulletin of the Calcutta Mathematical Society (BCMS), 21*, p.1-60, 1929.

dingly he examined the manuscript thoroughly and came to the conclusion that it was a work of 3rd to 4th century A.D.

The work contains rules, with illustrative examples and solutions of many arithmetical, algebraic and geometrical (including mensuration) problems. The major portion deals with arithmetic e.g. fractions, square roots, progressions, income and expenditure, profit and loss, computation of gold, interest, rule of three and summation of complex series. The discussion of algebraic operations includes simple equations, simultaneous linear equations, quadratic equations, problems relating to mensuration and miscellaneous subjects.

ĀRYABHAṬA I (c. 476 A.D.)

Āryabhaṭa I perhaps completed his education at the school of Kusumpura near modern Patna[1]. His *Āryabhaṭīya*[2] or *Āryasiddhānta (laghu)* which is complete in four chapters, namely, *Daśagītikā* (the ten Gīti stanzas), *Gaṇitapāda* (Mathematics), *Kālakriyā* (reckoning of time), and *Gola* (sphere), deals with arithmetic and astronomy. The work was written in 499 A.D. when he was 23 years old[3]. Thirty-

1. *Āryabhaṭīya*, introductory verse, Gaṇitapāda.
2. Ed. by H. Kern with commentary *Bhaṭadīpikā* of Parameśvara, Leiden, 1874; by Rodet, L. *Journal Asiatique*, *13*, p. 393-434, 1879; Tr. into English by P. C. Sengupta, *JDL, CU, 16*, p. 1-56, 1927; Tr. by W. E. Clark into English, Chicago, 1930; Ed. with commentary of Nīlakaṇṭha Somasutvan, *Trivandrum Sanskrit Series (TSS)*, nos. 101, 110 and 185, Trivandrum, 1931, 1932, 1957 respectively.
3. *ṣaṣṭhyabdānāṃ ṣaṣṭhiryadā vyatītāstrayaśca yugapādāḥ/*
tryadhikā viṃśatirabdāstadeha mama janmano'tītāḥ//
(*Āryabhaṭīya*, Kālakriyā v.10)

four verses of the *Āryasiddhānta* on astronomical instruments have recently been traced by K.S. Shukla in a commentery of the *Suryasiddhānta* by Rāmkṛṣṇa Ārādhya (1472 A.D.). Since these passages and Āryabhaṭa's midnight astromony do not occur in the *Āryasiddhānta* (laghu) of Āryabhaṭa, it is strongly presumed that he wrote another bigger work under the title, *Āryasiddhānta* (*mahā*). Though this work is not available at present, the glimpses of his midnight astronomy is still available in Bhāskara's *Mahābhāskarīya*, *Laghubhāskarīya* and in others. Some scholars believe that Āryabhaṭa I's works were based on the old *Sūryasiddhānta* for he was a worshipper of Sun, the promulgator of original *Sūryasiddhānta* (vide Introductory verse, Kern's edition)

The *Āryabhaṭīya* is essentially a systematisation of the results contained in the *Siddhāntas* and it is of particular value because of the picture it gives of the state of mathematical knowledge of the period, no less than for the impulse it gives to the study of the subject. In its pages are to be found alphabet-numeral system of notation, rules for the usual arithmetical operations including involution and evolution. He gave methods of solution of simple and quadratic equations, indeterminate equations of first degree. Āryabhaṭa I devoted considerable attention to trigonometry and his introduction of the sines and versed sines was a notable improvement upon the clumpsy half chords of Ptolemy. He hit upon a remarkably close approximations to the ratio of the circumference of a circle to its diameter (i.e., $\pi=3.1416$), which is

"when three Yugapādas and sixty times sixty years (3600 years) have elapsed then twenty-three years of my life have passed"—(Clark). The fourth Yugapāda of Āryabhaṭa I undoubtedly an achievement over the mathematicians of the world. He also gave correct generalised rules for computing the sum of natural numbers and of their squares, and cubes.

VARĀHAMIHIRA (c. 5(5–587 A.D.)

Varāhamihira was a native of Avanti and pupil of his astronomer father Ādityadāsa[1]. The statement of Utpala that he was a Magadha Brāhmaṇa must be understood in this sense that his family derived its origin from Magadha[2]. At the enquiry of Hunter[3], Hindu astronomers of Ujjain furnished a list assigning the date of Varāhamihira to be 427 Śaka era corresponding to 505 A.D. Alberuni[4] about eight centuries earlier from now gave the date which gives a corroborating evidence, though we do not know the exact period of his life. The date of Varāhamihira's death has been ascertained by Bhau Dāji[5] in a comm.

began with the beginning of Kaliyuga *i. e.* at 3102 B.C. Evidently in 499 A.D. (for 3600-3101=499), 23 years of Āryabhaṭa's life has passed.

1. *ādityadāsatanayastadavāptabodhaḥ kāpitthake*
   savitṛlabdhavaraprasādaḥ /
   avantiko munimatānyavalokya samyag ghorāṃ
   varāhāmihiro rucirāṃ cakāra //
   (*Bṛhajjātaka* ch. 26, v. 5)

   "Varāhamihira, a native of Avanti, was the son of Ādityadāsa and instructed by him. Having obtained the gracious favour of the sun, at Kapitthaka, composed his elegant work on horoscopy, after making himself duly acquainted with the doctrines of the ancient sages".
2. c/o. Colebrooke, Algebra, p. XLV, fn.
3. Published by Colebrooke, vide his *Miscellaneous Essays, 2,* p. 461, London, 1837. Also vide Bhaudaji's article, *Journal of the Royal Asiatic Society (JRAS,)* (n.s.), *1*, p.407, 1865.
4. *Journal Asiatique*, tome IV, p. 285, 1844.
5. "Varāhamihira went to heaven in the 509 in (year) of Śaka era i. e., 587 A.D." *JRAS, 1,* (n.s.), p.407, 1865.

of Āmarāja on the *Khaṇḍakhādyaka* of Brahmagupta to be 587 A.D.

Varāhamihira's *Pañcasiddhāntikā*[1], among other works[2], is considered important in the history of astronomy, as it gives the description of the five *siddhāntas* namely, Pauliśa, Romaka, Vaśiṣṭa, Saura and Paitāmaha of earlier dates. In the history of mathematics also, the work has a high place for its amount of trigonometrical information. It gives different relations among three functions, *jyā*, *kojyā* and *utkramajyā* and values of different *jyās* in a quadrant, drawn at a fixed interval (sine table) besides many other information of mathematical importance.

BHĀSKARA I (c. 600 A.D.)

We do not get sufficient evidence to say anything definitely about the place of his birth. In his *Āryabhaṭīyabhāṣya*, while giving problems on time and distance, he used the name of places like, Valabhi, Harkkaccha and Sivabhagapura. He gave the equinoxial shadow of Sthāneśvara. It partly indicates that he belonged to the part of India near Valabhi. It is known that he imbibed his knowledge of astronomy from his father. Modern researches have shown that Bhāskara I was not a direct pupil of Āryabhaṭa I, but was undoubtedly the most competent exponent of Āryabhaṭa I's school of Astronomy[3]. Kuppanna

1. Ed. with English translation by G. Thibaut and M. M. Sudhākara Dvivedī, Benaras, 1889; Reprinted by Motilal Banarsidas, Benaras, 1930; vide *Jyautiṣasiddhāntasaṃgraha* Ed. by V. P. Dvivedi, Benaras, 1917; Ed. and tran. into English by Neugebaner and Pingree, Kopenhagen 1970
2. *Bṛhatsaṃhitā*, *Bṛhajjātaka*, and some astrological works.
3. *IHQ*, *6*, p. 927-36, 1930.

Sastri[1] has concluded from external sources that Bhāskara I flourished in c. 600 A.D.

Three works were written by him in the following chronological order : (i) *Mahābhāskarīya*[2], an astronomical work complete in eight chapters, (ii) *Āryabhaṭīyabhāṣya*[3], a commentary on the *Āryabhaṭīya* of Āryabhaṭa and (iii) *Laghubhāskarīya*[4], an abridged and simplified version of *Mahābhāskarīya* in eight chapters. Bhāskara I was mainly an astronomer but made commendable progress in the solution of the indeterminate equation of the first degree, the method of whose solution is given in his *Mahābhāskarīya*[5] for use in the solution of astronomical problems[6]. Brahmagupta and others simply adopted the methods of Āryabhaṭa I and Bhāskara I. The method was however improved at a later period by Āryabhaṭa

1. Vide his edition of *Mahābhāskarīya*, Introduction, p. XII.
2. Ed. by Balavantaraya Apte with the commentary *Karmadīpikā* by Parameśvara, *Ānandāśrama Sanskrit Series* no. 126, Poona, 1945; Ed. with introduction and appendices by T. S. Kupanna Sastri, *Madras Govt. Oriental Series* no. 130, Madras 1957; Ed. with English translation and notes by Kripa Shankar Shukla, Lucknow University, 1960.
3. Ms; For Mss. vide *A Bibliography of Sanskrit Works on Astronomy and Mathematics*, by S. N. Sen, A. K. Bag and S. R. Sarma, National Institute of Sciences of India, New Delhi, p. 19, 1966. From now on, this bibliography will be abbreviated as our *Bibliography*.
4. Ed. with the commentary *Vivaraṇa* of Śaṅkaranārāyaṇa, Trivandrum Sanskrit Series no.162, Trivandrum, 1949; Ed. and translated into English by Kripa Shankar Shukla, Lucknow Univ. 1963.
5. Ch. 1, verses 41-52.
6. Examples are given in his *Mahābhāskarīya* (Ch. 8, verses 13, 19, 20, 23), *Āryabhaṭīyabhāṣya* (Ch. 2, verses 32-33), and *Laghubhāṣkrīya* (Ch. 8, verse 17).

II (c. 950 A.D.) by abridging the operations in certain cases.

BRAHMAGUPTA (c. 628 A.D.)

The most prominent of Hindu mathematicians belonging to the school of Ujjain was Brahmagupta. His father's name was Jiṣṇu. He wrote in 628 A.D. his *Brāhmasphuṭasiddhānta* at the age of thirty[1] and the other astronomical work, *Khaṇḍakhādyaka* in 655 A.D.[2] He lived during the reign of Śrīvyāghramukha, the greatest king of the Cāpa dynasty.

The *Brāhmasphuṭasiddhānta*[3] which is complete in twentyfour chapters includes two chapters on mathematics namely; *Gaṇitādhyāya* (chapter on mathematics) and *Kuṭṭakādhyāya* (chapter on indeterminate equations). Many other details of mathematical interests have been found in other chap-

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1. *śrīcāpavaṃśatilake śrīvyāghramukhe nṛpe śakanṛpālāt|*
   pañcāśatsaṃyaktairvarṣaśataiḥ pañcabhiratītaiḥ||
   brāhmaḥsphuṭasiddhāntaḥ sajjanagaṇitagolavit prītyai|
   triṃśadvarṣeṇa kṛto jiṣṇusutabrahmaguptena||
   (*Brāhmasphuṭasiddhānta, Ch. 24, v. 7-9*)
   "In the reign of Śrī Vyāghramukha, of the Srī Cāpa dynasty, five hundred and fifty years after the Śaka king (i.e., 628 A.D. having passed, Brahmagupta, the son of Jiṣṇu, at the age of thirty, composed the *Brāhmasphuṭasiddhānta*, for the gratification of mathematicians and astronomers".
2. Vide. P. C. Sengupta's tr. of the *Khaṇḍakhādyaka*, Introduction, p. XXVII, University of Calcutta, 1934.
3. Ed. with commentary of Sudhākara Dvivedi, Benaras, 1902; Published originally in the *Pandit*, (n.s.) *23-24*, 1901-2; tr. into English (a portion only) by H. T. Colebrooke, vide his *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupte* and *Bhascara*, London, 1817; Ed. with *Vāsanābhāṣya* by Ram Swarup Sarma and a board of editors and with notes by Satya Prakash. 4 vols. Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.

ters of this work. The *Gaṇitādhyāya* deals with cyclic triangle and quadrilateral, rules for arithmetical operations involving zero, negative numbers, quadratic equations. The *Kuṭṭakādhyāya*, contains solutions of the indeterminate equations of both first and second degree. The *Spaṣṭādhikāra* chapter contains, however, trigonometrical notations including sine table. In chapter IX of his *Khaṇḍakhādyaka*, Brahmagupta gave a method of obtaining from the given table of sines, the sines of intermediate angles.

Brahmagupta was a bitter opponent of Āryabhaṭa I, though in his *Khaṇḍakhādyaka*, in his old days he seems to have recognised Āryabhaṭa's merit and adopted one of the system of Āryabhaṭa's astronomy. Brahmagupta holds a place in the history of Indian Civilisation. His works which were translated into Arabic seem to have influenced Arabian astronomy and mathematics in the 8th century A.D. possibly before Arabian scholars came to know about Ptolemy's work.[1]

LALLA (c. 768 A.D.[2])

Lalla was the grandson of Śāmba and son of Bhaṭṭa Trivikrama. His *Śiṣyadhivṛddhida*[3] in one thousand ślokas is fully devoted to astronomy. The work contains some important information on trigonometry. He was associated with the school of Kusumpura and

1. Arabian *Sindhind* and *Alarkand* were probably translations of *Brāhmasphuṭasiddhānta* and *Khaṇḍakhādyaka*. According to some scholars, *Sindhind* may be a translation of the *Sūrya-siddhānta*, or some other work bearing the title *siddhānta* (Smith, D. E. History of Math., *1*, p. 167-8. Dover Publication).
2. Sengupta, P. C. *Khaṇḍakhādyaka*, Introduction, p. XXVI, Calcutta, 1934.
3. Ed. by Sudhākara Dvivedi, Benares, 1886.

his work was based on the *Āryabhaṭīya* of Āryabhaṭa I. Lalla has written two other works namely, *Pāṭīgaṇita* and *Siddhāntatilaka* both of them are now lost. Only references are available in later works.[1] The *Pāṭīgaṇita* dealt exclusively with mathematics, whereas the *Siddhāntatilaka* which was similar to the *Brāhmasphuṭa siddhānta* contained chapters on arithmetic and algebra.

GOVINDASVĀMIN (c. 800-850 A.D.)

Govindasvāmin who wrote a *bhāṣya* on the *Mahābhāskarīya*[2] of Bhāskara I seems to have flourished in Kerala in the first half of the ninth century A.D. Nīlakaṇṭha refers to the tradition that Govindasvāmin was the teacher of Śaṅkaranārāyaṇa who flourished at Mahodayapura. His *Govindākṛti* which was written as a sequel to the *Āryabhaṭīya* of Āryabhaṭa I appears to be lost. It was also similar to the *Brāhmasphuṭasiddhānta* of Brahmagupta and contained chapters on arithmetic and mensuration. Quotations from this work are found to appear in the commentaries of the astronomers, Śaṅkaranārāyaṇa (869 A.D.) and Udayadivākara (1073 A.D.) on the *Laghubhāskarīya*[3] of Bhāskara I and in the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha.

SKANDASENA (c. 9th century beginning)

Pṛthudakasvāmi (860 A.D.) has referred to this

1. Vide *Pāṭīgaṇita* of Śrīdharācārya. Ed. by K. S. Shukla, Introduction, p. IX, Lucknow Univ. 1957.
2. Ed. by T. S. Kuppanna Śāstri with the *Bhāṣya* of Govindasvāmin and *Siddhāntadīpikā* of Parameśvara, Madras, 1957; vide. our *Bibliography*, p. 78.
3. Vide the *Laghubhāskarīya* with the commentary *Vivaraṇa* of Śaṅkaranārāyaṇa, Trivandrum Sanskrit Series no. 162, Trivandrum, 1949; For Mss. of Udayadivākara vide *Bibliography*, p. 230.

mathematician three times in his commentary on ch. 12 of the *Brāhmasphuṭasiddhānta* of Brahmagupta (v.2, 9 and 55). This reference to Skandasena is of great historical importance, as they show that the methods of multiplication known as *kapāṭasandhi*, *tatstha* and *khaṇḍa* as well as the mixed fractions called *bhāgamātā* occurred also in early works on Hindu *pāṭīgaṇita*. We also understand that the geometrical interpretation of an arithmetic series with the help of a figure which has been explained in detail in Śrīdhara's *Pāṭīgaṇita* (vide Rules 79-89) is due to Skandasena or some earlier mathematician and not to Śrīdhara. The works of Skandasena are now lost and nothing more can be said definitely (vide Shukla's Introduction of the *Pāṭīgaṇita* of *Śrīdhara*, Lucknow University, 1959).

PṚTHUDAKA SVĀMI (c. 850 A.D.)

Pṛthudakasvāmi wrote in 860 A.D. an important commentary[1] on the *Brāhmasphuṭasiddhānta*, which contains many examples to illustrate the mathematical statement and results contained in the *Brāhmasphuṭasiddhānta.* A considerable difference of opinion prevails whether these examples are those of Brahmagupta or of Pṛthudakasvāmi. The examples are not found in any text of the *Brāhmasphuṭasiddhānta.* The writing materials were scarce in those days and the students learnt at the feet of their masters and passed that knowledge in turn to their students. It may be that in this way some or all the examples might have come down to Pṛthudakasvāmi.

1. Vide. Colebrooke's translation with comm. of Pṛthudakaswāmi in his *Algebra with Arithmatic.........from the Sanskrit of Brahmagupte and Bhāscara*, London, 1817.

MAHĀVĪRA (c. 850 A.D.)

The fame of Mahāvīra, a Jaina mathematician rests on his brilliant work known as the *Gaṇitasārasaṃgraha*[1]. This was written in 850 A.D. From the comparison of his work and that of Brahmagupta, it seems that Mahāvīra was familiar with latter's work but he has improved upon it a lot. He had associations with the school of Mysore and lived perhaps at the court of Amoghavarṣa Nṛpatunga, one of the Rāṣṭrakuṭa monarchs in Mysore.

The work in nine chapters is exclusively written on arithmetic, geometry and algebra. He has dealt with almost all the problems of his predecessors. He made the classification of the arithmetical operations simpler and gave a number of examples to elucidate the rules. His treatment of the work appears quite lucid and elaborate. His most noteworthy work however is the treatment of fractions. Mahāvīra shows ability to handle geometrical as well as arithmetical series. He gives a general rule of the combination of n different things taken r at a time. He deals with different types of quadratic equations and with the right angled triangle whose sides are rational. He continued on the line of his predecessor in dealing with indeterminate equation and fundamental arithmetical operations.

ŚRĪDHARA (c. 850-950 A.D.)

Śrīdharācārya's *Pāṭīgaṇita*[2] is a work on arithme-

1. Translated into Telugu by Pabuturi, 11 th century A. D., Edited and tr. into English with notes by M. Raṅgācārya, Madras, 1912.
2. Ed. by Sudhākara Dvivedi, Nirnayasagar Press, 1879; Ed. with English translation and notes by Kripa Shankar Shukla, Lucknow University, 1959.

tic and mensuration. He dealt with multiplication, division, square, cube, square-root, cube-root, fraction, rule of three and the areas of plane figures. Eight rules are given for operation involving zero but division by zero is not one of them. His method of summation of different A.P. and G.P. and solution of the problems pertaining to mixtures of things are interesting. He for the first time gave a rule to extract the root of $ax^2+bx=c$, which is known usually as Śrīdhara's formula.

There remains a great deal of controversy about his time and place of origin. Some scholar opined that he was a Ācārya Brāhmin of Bengal and a Saivite Hindu, and others assigned him to the South India. Neither of these views are considered tenable by Shukla.[1] From a comparative study of the works of Śrīdharācārya with those of Mahāvīra (850 A.D.) and Āryabhaṭa II (950 A.D.), he came to the conclusion that Śrīdharācārya flourished between c. 850 A.D. to 950 A.D.

ĀRYABHAṬA II (c. 950 A.D.)

The *Mahābhāskarīya*[2], an astronomical work in eighteen chapters of Āryabhaṭa II dealt various problems of mathematical interest, besides preliminary operations. The work was written in 950 A.D. Āryabhaṭa II mentions separately about three branches of mathematics viz. *pāṭī*, *kuṭṭaka* and *bīja* in his *Mahāsiddhānta* (ch. 1,v.1). Chapter XV of his treatise contains the *pāṭīgaṇita* and chapter XVIII deals with

1. Vide his edition of *Pāṭīgaṇita* of Śrīdhara, Introduction, p. XXXV-XLIII.
2. Ed. with his own commentary by Sudhākara Dvivedi, *Benares Sanskrit Series* no. 36, Benares, 1910.

kuṭṭaka. In solving the quadratic, he followed the method of Āryabhaṭa I and Brahmagupta. He advised some corrections in the treatment of solution of simultaneous indeterminate equations of first degree. The high estimation of the subject of *kuṭṭaka* continued so distinctly after Āryabhaṭa II that the subject received special treatment at the hands of Hindu workers after Āryabhaṭa II onwards.

ŚRĪPATI (c. 1039 A.D.)

Śrīpati was a Jaina astronomer cum mathematician. He wrote *Gaṇitatilaka*[1], *Siddhāntaśekhara*[2] (in 1039 A D.) and *Bījagaṇita* besides five other works on astronomy and astrology[3]. The *Gaṇitatilaka* is devoted exclusively to arithmetic. The *Siddhāntaśekhara*, a work mainly on astronomy in twenty chapters, deals with algebra in two chapters namely *vyaktagaṇitādhyāya* (ch. 13) and *avyaktagaṇitādhyāya* (ch. 14). But his *Bījagaṇita* is now lost. Though he is the author of so many works, very little is known about his life. Probably he was the son of Nāgadeva and was a native of Rohinikhaṇḍa[4]. Like his predecessors, he discussed in his *Gaṇitatilaka* the eight fundamental operations and dealt with permutation, combination, proportion, notational places and many other things of relative interst.

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1. Ed. with introduction by H. R. Kapadia with comm. of Siṃhatilaka Suri. *Gaekwad Oriental Series* no. 78, Baroda, 1937.
2. Ed. by Babūaji Miśra, 2 vols. Calcutta University, 1936, and 1947.
3. *Dhikoṭikaraṇa*, *Jyautiṣaratnamālā*, *Dhruvamānasa*, *Jātakapaddhati* and *Daivajñavallabha*.
4. Vide. *Gaṇitatilaka* introduction, by H. R. Kapadia.

BHĀSKARA II (c. 1114 A.D. to 1200 A.D.)

The general appreciation and fame of Bhāskara II or Bhāskarācārya, as he was popularly called as a mathematician, is based on his *Līlāvatī*[1], *Bījagaṇita*[2] and *Siddhāntaśiromaṇi*[3]. He is the author of two other works namely *Vāsanābhāaṣya*, his own commentary on the *Siddhāntaśiromaṇi*, and *Karaṇakutūhala*[4], a treatise on planetary motion. As to the time of his

1. Tr. by J. Taylor, Bombay, 1817; Tr. by H. T. Colebrooke (a portion only) vide his *Algebra with Arithmetic............*, London, 1817; Ed. by Bāpudeva Śāstrī, Benares, 1883; Ed. by Sudhākara Dvivedi, Benares, 1912; Colebrooke's translation with notes and texts by Haran Candra Banerjee, Calcutta 1893; second edition, Calcutta, 1927; Ed. by Dattatreya Apte with *Buddhivilāsinī* comm. of Gaṇeśa Daivajña and *Līlāvatīvilāsa* of Mahīdhara, Ānandāśram Sanskrit Series no. 107, Poona, 1937; Ed. by Payanātha Jhā with commentary *Vāsanā* by Dāmodara Miśra, Mithila Institute of Post Graduate Studies in Sans. Learning, Durbhanga, 1959 etc.
2. Tr. by E. Strachey, London, 1813; By H. T. Colebrooke (a portion only) vide his *Algebra with Arithmetic.........*, London, 1817; Ed. by Bāpudeva Śāstrī, 2 parts, 1875; Ed. with comm. by Sudhākara Dvivedi, Benares, 1888; Ed. with commentary *Navāṅkura* of Kṛṣṇa Daivajña, Ānandāśrama Sanskrit Series no. 99, Poona, 1930; Ed. by T. V. Rādhākṛṣṇa Śāstri with comm. *Vījapallava* of Kṛṣṇa Daivajña, Tanjore Saraswati Mahal Series no. 98, Tanjore, 1958 etc.
3. Ed. and trans into Eng. (Golādhyāya only) by L. Wilkinson with the *Vāsanābhāṣya*, Calcutta, 1842; Revised edition by Bāpudeva Śāstri, Calcutta, 1861; Ed. by Jīvānanda Bhattācārya with *Mitākṣarā* of Raṅganātha, Calcutta, 1881; Ed. by Muralidhara Jhā with *Vāsanābhāṣya*, *Vāsanāvārttika* of Nṛsiṃha and *Marīci* of Viśvarūpa, Benares, 1917; Ed. by Bāpudeva Śāstrī with *Vāsanābhāṣya*, revised by Ganapati Śāstrī, Kāśi Sans. Series no. 72, Benares, 1929, etc.
4. Ed. by Sudhākara Dvivedi with comm. *Vāsanā*, Benares, 1881; Ed. with comm. of Sumatiharṣa, Bombay, 1901.

activity, we gather the following statement from his own writing, "In the year 1036 of the Śaka kings I was born, and at the age of 36 I have written the *Siddhāntaśiromaṇi*."[1] The year of his birth i.e., 1036 of the Śaka kings corrosponds to 1114 A.D. and the *Siddhāntaśiromaṇi* was written in 1150 A.D. The *Karaṇakutūhala* was written in 1183 A.D. Of his parentage and native place, he has said that he was born of a renowned Brāhmaṇa scholar and astronomer Maheśvara at a city called Bijjaḷabiḍa.[2] He had association with the schools of Ujjain and received his education from his father. All these have been confirmed by an inscription discovered by Bhau Daji.[3]

The *Līlāvatī* which is based on Brahmagupta's *Brāhmasphuṭasiddhānta*, Śrīdhara's *Pāṭīgaṇita* and Āryabhaṭa II's *Mahāsiddhānta* exhibits a profound system of arithmetic and also contains many useful propositions in geometry and arithmetic. In addition to fundamental operations his rule of three, five, seven, nine, and eleven, permutation and combination, treatment of zero prove his deep knowledge in the subject. The work was translated into Persian by Fyzi in 1587 A.D. by the direction of the emperor

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1. *rasaguṇapūrṇamahīsamaśakanṛpasamaye'bhavanmamotpattiḥ|*
   rasaguṇavarṣena siddhāntaśiromaṇi racitāḥ||
   (vide Bāpudeva Śāstrī's ed. of the *Siddhāntaśiromaṇi*, *Gola*, XIII, 58)
2. Ibid. Gola, XIII, 61-2. Compare also the concluding verse of the *Bījagaṇita*. In 1150 A.D. when Bhāskara wrote his *Siddhāntaśiromaṇi*, the town of Biḍa was under the rule of Prince Bijjala, a vassal of the Western Cālukya King Tailapa II and hence it has been called Bijjala-Biḍa.
3. *JRAS*, *1*, (n.s.), p. 414-418, 1865.

Akbar. A curious account of the occasion of writing the *Līlāvatī* is current which is given by Fyzi[1] in the preface of his translation. His *Bījagaṇita* contains problems of unknown quantities, surds, pulverizer (or *kuṭṭaka*), simple and quadratic equations including general rule which went beyond that of Śrīdhara, solutions of several indeterminate equations of second degree and solutions of certain equations of the third and fourth degree. The full solution of the equation and of its more general form $ax^2+bx+c=y^2$ was given by Bhāskara II. About his 'Cyclic method' Hankel[2] wrote, "It is beyond all praise : it is certainly the finest thing achieved in the theory of numbers before Lagrange." H.T. Colebrooke[3] points out that the method is exactly the same which Lord Brouncker devised to answer a question proposed by way of challenge by Fermat in 1657. In his *Siddhāntaśiromaṇi*, we get evidence of his knowledge of trigono-

1. It is said that *Līlāvatī* was the name of Bhāskara II's daughter. Bhāskara II came to know by the power of his astrological knowledge that she should never be wed. The father, however, ascertained a lucky hour for contracting her in marriage. He placed a cup with a small hole at the bottom on a vessel of water. It was so arranged that the cup would sink at the end of an hour. When everything was ready and the cup was placed on the vessel of water, Līlāvatī suddenly out of her childish curiosity came near to the cup to observe the water coming into the cup when, by chance, a pearl being separated from her bridal dress, fell into the cup and chanced to stop the influx. The lucky hour passed without the sinking of the cup. Bhāskara II, out of great dejection wrote a book and named it *Līlāvatī* after her name.
2. Hankel, H. *Zur Geschichte der Mathematik*, p. 202, Leipzig, 1874.
3. *Bījagaṇita*. Rules 80-81 (vide his *Algebra with Arithmetic....* London, 1817.

metry including sine table and different relations among the three functions known as *jyā*, *kojyā* and *utkramajyā*. The idea of differentiation of the earlier workers was finally developed by him. He was acquainted with the principle of infinitesimal calculus and is often given credit for originating the idea of integration long before Newton and Leibniz.

He earned so great a distinction that his manuscripts have been copied and commented widely in later years. An ancient temple inscription referred to him in the following terms : "Triumphant is the illustrious Bhāskarācārya whose feet are revered by the wise, eminently learned......, a poet,..... endowed with good fame and religious merit..............."[1]

4. LATE MEDIEVAL PERIOD—A Period of Commentaries and Some New Developments (c. 1200 A.D. to 1800 A.D.)

In this period, Bhāskara II's works became so popular that the scholars of North India were perhaps content with the preservation and transmission of the knowledge from generation to generation. Several hundreds of commentaries were written on the *Līlāvatī*, *Bījagaṇita*, and *Siddhāntaśiromaṇi*. Copies of large number of these manuscripts are now available in India and abroad. For giving a picture, to what extent Bhāskara II's works gained popularity, a list of commentaries on the *Līlāvatī* is appended here as an illustration.

1. Kaye, G. R. *Indian Mathematics*, p. 37, Calcutta, 1915.

Serial No.	Name of the commentators on the *Līlāvatī* of Bhāskarācārya (c. 1150 A.D.)	Approximate time of these commentators	Name of the commentaries
1.	Gaṅgādhara	c. 1420 A.D.	*Gaṇitāmṛtasāgarī*
2.	Parameśvara	c. 1430 A.D.	*Līlāvatīvyākhyā*
3.	Moṣadeva	c. 1473 A.D.	*Līlāvatīṭīkā*
4.	Gaṇeśa Daivajña	c. 1545 A.D.	*Buddhivilāsinī*
5.	Dhaneśvara	earlier than 1541 A.D.	*Līlāvatībhūṣaṇa*
6.	Sūryadāsa	1541 A.D.	*Gaṇitāmṛtakūpikā*
7.	Nārāyaṇa	earlier than 1588 A.D.	*Karmapradīpaka*
8.	Anonymous	earlier than 1609 A.D.	*Līlāvatīprakāśa*
9.	Raṅganātha	1643 A.D.	*Mitabhāṣiṇī*
10.	Rāmakṛṣṇa	c. 1650 A.D.	*Gaṇitamṛtalaharī*
11.	Vireśvara	16th century	*Pāṭīvyākhyāna*
12.	Parasurāma	earlier than 1659 A.D.	*Līlāvatīvivaraṇa*
13.	Keśava	earlier than 1711 A.D.	*Līlāvatīvyākhyā*
14.	Śrīdhara Mahāpātra	c. 1717 A.D.	*Sarvabodhinīvyākhyā*

Serial No.	Name of the commentators on the *Līlāvatī* of Bhāskarācārya (c. 1150 A.D.)	Approximate time of these commentators	Name of the commentaries
15.	Mahīdāsa	c. 1722 A.D.	*Līlāvatīvivaraṇa*
16.	Dāmodara	earlier than 1748 A.D.	*Pāṭīlīlāvatīṭīkā*
17.	Rāmakṛṣṇa Deva	earlier than 1750 A.D.	*Manorañjanā*
18.	Mahīdhara	earlier than 1755 A.D.	*Līlāvatīvivaraṇa*
19.	Devīsahāya	before 1760 A.D.	*Līlāvatīvilāsa*
20.	Anonymous	earlier than 1766 A.D.	*Bījapāṭī*
21.	Sūryamaṇi	before 1800 A.D.	*Gaṇitāmṛtavarṣiṇī*
22.	CandraŚekharaPaṭanāyaka	befor 1818 A.D.	*Līlāvatyudāharaṇa*
23.	Kāma	earlier than 1800 A.D.	*Līlāvatīvyākhyā*
24.	Nīlāmbara jhā	c. 1825 A.D.	*Līlāvatyudāharaṇa*
25.	Ahosaka Deva	—	*Līlāvatīvyākhyā*
26.	Rāma Candra	—	*Gaṇitāmṛtakūpikā*
27.	Rāmeśvara Miśra	—	*Līlāvatīṭīkā*
28.	Śrīkṛṣṇa	—	*Līlāvatīṭīkā*
29.	Viśveśvara	—	*Līlāvatyudāharaṇa*

Serial No.	Name of the commentators on the *Līlāvatī* of Bhāskarācārya (c. 1150 A D.)	Approximate time of these commentators	Name of the commentaries
30.	Vṛndavana Śukla	c. 1825 A.D.	*Līlāvatīṭīkā*
31.	Mādhava	—	*Karmapradīpikā*
32.	Anonymous	—	*Kūpikā*
33.	„	—	*Laghuvimarṣiṇī*
34.	„	—	*Līlāvatīkaṭākṣa*
35.	„	—	*Līlāvatīṭīkā*
36.	„	—	*Līlavātī udāharaṇa*
37.	„	—	*Līlāvatīvilāsa*
38.	„	—	*Līlāvatīvivaraṇa*
39.	„	—	*Līlāvatīvivṛti*
40.	„	—	*Līlāvatīvyākhyā*
41.	„	—	*Pāṭīlīlāvatīvṛtti*

On the above commentaries of the *Līlāvatī*, there are altogether 143 copies of this manuscript now available (vide our *Bibliography*, p. 23-25).

To estimate the progress made by these commentators, some belonging to a family of astronomers,[1] details of the nature of their works need be given here.

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1. *Geneology :*

(a) *Jambusāranagara :*

Divākara — Govardhana — Gaṅgādhara
(c. 1375 A.D.) (c. 1400) (c. 1420)

(b) *Dadhigrāma :*

Rāma — Trimalla + Gopirāja
(c. 1425) (c. 1450)
Trimalla — Vallāla
(c. 1450)
Vallāla — Rāma + Kṛṣṇa + Govinda + Raṅganātha
(c. 1495) (c. 1520) (c. 1548) + Mahādeva
Govinda — Nārāyaṇa
(c. 1568) (1588)
Raṅganātha — Muniśvara
(1603)

(c) *Nandigrāma :*

Kamalākara Daivajña — Keśava — Gaṇeśa
(c. 1466 A. D.) (1496) (1507)
Keśava — Rāma — Nṛsiṃha
(1496) (1522) (1548)

(d) *Pārthapura :*

Nāganātha — Jñānarāja — Sūryadāsa
(c. 1480) (c. 1503) (c. 1541)

(e) *Golagrāma :*

Divākara Daivajña—Viṣṇu+Kṛṣṇa+Mallāri+Keśava
(c. 1500 A. D.) (c.1556) (1565) (1571)
+Viśvanātha
(1580)
Kṛṣṇa — Nṛsiṃha+Śiva
(1565) (1586)
Nṛsiṃha—Divākara+Kamalākara+Gopinātha+
(1586) (c. 1606) (c. 1616)
Raṅganātha
(1643)

Gaṅgādhara (c. 1420 A.D.) was the son of Govardhana and grandson of Divākara, an inhabitant of Jambusāranagara. His commentary, *Gaṇitāmṛta-sāgarī* or *Amṛtasāgarī* though confined to the *Līlāvatī* contains a chapter on the *Bījagaṇita*. This work appears almost verbatim of the original work.

Gaṇeśa (c. 1507 A.D.) was the son of Keśava Daivajña, a distinguished astronomer and a native of Nandīgrāma near Devagiri. His *Buddhivilāsinī* commentary on the *Līlāvatī* was written in 1545 A.D. It gives a copious exposition of the text with proper demonstration of the rules. It can be considered as one of the best among available commentaries.

Sūryadāsa (c. 1541 A.D.), was the son of Jñānarāja and native of Pārthapura, a village near the confluence of the Godavari and the Vidarbha rivers. His gloss on the *Līlāvatī* and the *Bījagaṇita* are known as *Gaṇitāmṛtakūpikā* (or *Amṛtakūpikā*) and *Sūryaprakāśa Bījavyākhyā* respectively. Both contain excellent clear interpretation of the text with a concise explanation of the principles of the rules.

Kṛṣṇa (c. 1548 A.D.) of Dadhigrāma was son of Vallāla and disciple of Viṣṇu. His *Bījapallava* (or *Kalpalatāvatāra*), a commentary on the *Bījagaṇita* of Bhāskarācārya contains a clear and copious exposition of the sense with ample demonstration of the rules much in the manner of Gaṇeśa.

Raṅganātha (c. 1643 A.D.) of Golagrām was the son of Nṛsiṃha Daivajña and grandson of Kṛṣṇa. His commentary on the *Līlāvatī*, entitled *Mitabhāṣiṇī* demonstrates the original passages with many demonstrations.

Rāmakṛṣṇa was the son of Lakṣmīdāsa and grandson of Nṛsiṃha. His *Bījaprabodha*, commentary on the *Bījagaṇita* of Bhāskara II and *Gaṇitāmṛtalaharī*, a commentary on the *Līlāvatī* of Bhāskara II are similar to Raṅganātha's work.

Rāmakṛṣṇa Deva, was the son of Sadāśiva Āpadeva. His *Manorañjanā*, a commentary on the *Līlāvatī* of Bhāskara II is a copious exposition of the text with demonstrations here and there and so on.

Though it has not become possible to turn up all the manuscripts yet our first hand checking of the contents suggest that no notable progress of arithmetic, algebra and geometry were made by these commentators.

Such works gave a new impetus to the studies in Kerala and from fourteenth century onwards, there has been an unbroken tradition for about five hundred years (*vide*, p. 45). The mathematicians of Kerala were staunch followers of Āryabhaṭa I's school, but they understood the significance and limitations of the previous scholars.

Kaṭapayādi system[1].—In this connection, it would not be quite out of place if we discuss a system of numerical notation namely, *Kaṭapayādi*, a system known in South India from 6th century A.D. from the time of Haridatta. This had begun to be utilised widely after Bhāskara II in South Indian treatises to express the time of the authors, *ahargaṇas*, eclipses and other related matters. The system[2] is almost self explanatory : *Kaṭapayādi* means starting with 'k, ṭ, p and y'. The letters *k* to *jh* indicate 1 to 9 respecti-

1. Another system known as *bhūtasaṃkhyā* was current in South India. Here the numbers were indicated by well-known objects. For examples : 'eyes' or 'hands' indicate 2; 'senses' or 'elements' indicate 5. For details, vide Datta & Singh, *Hist. of Hindu Maths.*, *1*, p. 69.
2. Bag, A. K. 'Trigonometrical Series in the *Karaṇapaddhati* and the date of the text', *Indian Journal of Hist. of Science*, *1*, No. 2, p. 98-106, 1966; K. Kunjunni Raja, *Adyar Library Bulletin*, *27*, p. 122-131, 1963.

vely; so also *ṭ* to *dh*; *p* to *m* stand for 1 to 5 and *y* to *h* represent 1 to 8 respectively; ñ and n and vowels not preceded by a consonant stand for zero. In case of conjunct consonants, it is the last consonant that has value. The vowels following a consonant have no special value. The letter *ḷ* peculiar to the Drabidian pronunciation of Sanskrit is given the value 9. In this system a right to left arrangement is employed. The system is beautifully explained in the *Sadratnamālā*.

The activities of the scholars of this period who attained an independent distinction have been described here separately. Many of their manuscripts still unpublished are yet to be published and studied.

NĀRĀYAṆA PAṆḌITA (c. 1356 A.D.)

Nārāyaṇa Daivajña or Nārāyaṇa Paṇḍita, as he was popularly called, was the son of Nṛsiṃha Daivajña. He is solely a mathematician and the author of two works namely (i) *Gaṇitakaumudī*[1], a mathematical treatise in 14 chapters and (ii) *Bījagaṇitāvataṃsa*[2], a work on algebra.

The date of composition of the *Gaṇitakaumadī*[3] is given in the work itself as *gaja-naga-ravi-mite* śake or 1278 śaka corresponding to 1356 A.D.

In this work he dealt, in addition to the eight fundamental operations, the problems on series and a general formula for the sum of any order of triangular numbers. Nārāyaṇa's method of summation of any order triangular number or *vārasaṃkalita* shortly, has established a way for the development of inte-

1. Ed. by Padmākara Dvivedi, two parts, The Prince of Wales Saraswati Bhavan Text No. 57, Benaras, 1936 and 1942.
2. See Algebra of Nārāyaṇa by B. Datta, *Isis*, *19*, 472-85, 1933; Ed. by K. S. Shukla, Lucknow, 1970.
3. *Ibid.*, Part II, p. 411; vide also the preface of *Gaṇitakaumudī* (Part I), *Saraswati Bhavan Text* no. 57, 1936.

gration series in connection with quadrature of the circle and allied problems. In his *Bījagaṇita*, besides usual topics on algebra, the method of finding the approximate values of the square-root of a non-square number is noteworthy.

MĀDHAVA (c. 1400 A.D.)

Mādhava, a Kerala astronomer is well-known for his *Veṇvāroha*[1]. He was regarded as the authority on spherical astronomy and mathematics and is often referred to by later writers as a *golavid*[2] (expert on spherical mathematics). Mādhava belonged to Saṃgamagrāma[2], a village of the name Saṃgama[3]. Mādhava gives also his family title as *bakula vihāram*[4] in his *Veṇvāroha* (verse 13).

Mādhava's date of writing *Veṇvāroha* is fixed up to be 1400 A D. from an epoch he used in this work[5]. This date can also be supported by other evidences. Pārameśvara, one of his disciple wrote his *Dṛggaṇita*

1. Ed. by K. V. Sarma with the Malayalam commentary of Acyuta Piṣāraṭi, *Ravivarma Sanskrta* Series No. 7, Trippunithura, 1956.
2. (a) *tacca saṃgamagrāmajena golatattvavida mādhavena pradarśitam* (*Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, TSS. 185, p. 75)
   (b) *vande golavidaśca mādhavamukhān* (*Sphuṭanirṇaya* by Acyuta Piṣārati, Oriental Library, Madras, Ms. R. 3799 (b)).
3. *Irriññālakkuṭa* in Cochin, famous for the temple of Saṃgameśvara.
4. Bakulam=Iraññi, Viharam=palli. So the local form of the word is *Iranni-ninna-palli*. This house exists even now in that place, though the present descendents have failed to preserve their scholarly tradition.
5. The epoch was used as 1502008 (*dinanamrānuśāsya*) Kali days together with 5180 (*ādikurma*) cycles of moon. The first numerical figure gives 1010 years and the second gives 390 years. Two together amount to 1400 A. D.

in 1431, A.D. (vide infra). Hence it is quite likely that Mādhava wrote thirty years before that of his disciple. Therefore Mādhava may be considered to have flourished during the end of fourteenth century.

Besides *Veṇvāroha* which deals with an accurate method of calculating the exact positions of the moon Mādhava is well-known for his work[1] on trigonometry and on infinite series with correct determination of the value of π. Nīlakaṇṭha quotes profusely Mādhava's name and says that he has used Mādhava's theorems in his *(Āryabhaṭīya) bhāṣya.* In one place he quotes a verse attributing it to Mādhava and admits that he has heard it from an astrologer while travelling in North Kerala[2]. This suggests that Mādhava besides *Veṇvāroha* might have composed other works which though extinct at present became popular throughout Kerala. Mādhava is largely quoted in later Malayalam works namely, the *Yuktibhāṣā* of Jyeṣṭhadeva, *Kriyakramakarī*, a commentary on the *Līlāvatī* etc.

PRRAMEŚVARA (c. 1430 A.D.)

Parameśvara, the founder of the *dṛggaṇita* system of astronomy in Kerala though mainly an astronomer was held with respect in the field of mathematics.

1. Vide. Ramavarma Maru Tampuran, '*vṛttaparidhiyiluṭe*' Mangalodayam, Vol. 20, p. 252-6; the verse giving the value of, *vyāse vāridhinihate rupahate*......is quoted as Mādhava's in the *Kriyākramakarī* Comm. on the *Līlāvatī* (Madras, R. 2754, p. 362). Vide also. K. V. Sarma's article on Mādhava, *Quarterly Journal of the Mythic Society, 49*, No. 3, p. 183-86, 1958.

2. *tadviṣayamapyanuṣṭubantaraṃ mādhavoktaṃ mūṣika deśajād daivajñāt paryaṭatā śrutaṃ mayā (muṣikadeśa* is Kolattunāḍ in North Kerala) *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, Vol. II, p. 47.

He wrote commentaries on all the popular classical works on astronomy and mathematics besides some original works on astronomy[1]. His commentaries are : the *Bhaṭadīpikā*[2] on the *Āryabhaṭīya*, the *Parameśvarī*[3] on the *Laghubhāskarīya* of Bhāskara I, the *Siddhāntadīpikā*[4] (or *Karmadīpikā*), a *bhāṣya* on the *Mahābhāskarīya*, the *Parameśvara*[5] on Muñjāla's *Laghumānasa*, the *Vivaraṇa*[6] on the *Sūryasiddhānta*, and the *Vyākhyā*[7] on the *Līlāvatī* etc.

Various information about Parameśvara is available from Nīlakaṇṭha's *Āryabhaṭīyabhāṣya* and from later works also. He belonged to the Alattūr village (Sanskritised as *aśvathagrāma*) and that his house was situated on the northern bank of the river Nīlā[8]. The name of his house Vataśśeri (Sanskritised as

1. *Dṛggaṇita* (Ed. K. V. Sarma, Hoshiarpur, 1963), *Goladīpikā* I (*TSS*. 49), *Goladīpikā* II (Ed. Adyar Library Pamphlet Series no. 32), etc. Vide. K. V. Sarma's article on Parameśvara, *Journal of the Oriental Research Institute*, Madras, *28*, Pts. i-iv, p. 47, 1961; *Adyar Library Bulletin*, *27*, 136-143, 1963; our *Bibliography*, National Institute of Sciences of India, p. 167-70, 1966.
2. Ed. by H. Kern, Leiden, 1874.
3. *Ānandāśrama Sanskrit Series*, no. 128. Another work of the same title is referred to by V. Rājarāja Varma, op. cit., *2*, p. 502.
4. *Madras Govt. Oriental Ms. Library*, No. 130.
5. *Ānandāśrama Sanskrit Series* No. 123.
6. Ed. by Kripa Shankar Shukla, Department of Mathematics and Astronomy, Lucknow University, 1957.
7. Vide. Our *Bibliography*, p. 169.
8. *nīlāyāḥ sāgarasyāpi tīrasthaḥ parameśvaraḥ /*
   vyākhyānamasmai bālāyā līlāvatyāḥ karomyaham //
   (Introductory verse of Parameśvara's Comm. on *Līlāvatī*, *Adyar* Library Ms. 68524). See also his Comm. on the *Sūryasiddhānta*. He has been referred sometimes as *Paramādiśvara*, but his name Parameśvara is also given frequently.

vaṭaśreṇi[1]) occurs in his name as usual in the Southern India. A meagre information is available about his parents. In his own work it is given that his grandfather was the pupil of Govinda (different from the author of *Mahābhāskarīyabhāṣya* who lived much earlier), the famous author of *Muhurtaratna.* He came of the Bhṛgugotra and belonged to the Āśvalāyana school.

Parameśvara flourished during the first half of the fifteenth century as recorded in his own works *Dṛggaṇita*[2] (1431 A.D.) and *Goladīpikā*[3] (1443 A.D.).

NĪLAKAṆṬHA SOMASUTVAN (c. 1443 to 1543 A.D.)

Nīlakaṇṭha[4] is mainly an astronomer but his *Āryabhaṭīyabhāṣya*[5] and *Tantrasaṃgraha*[6] contribute much to mathematics.

He was a Nambutiri Brāhmin of Kellalure family of Tṛ-k-kaṇṭiyur near Tirur, South Malabar. He

1. Vide. *Adyar Library Bulletin, 27*, p. 137, 1963.
2. *evaṃ dṛggaṇitaṃ śake, triṣuviśvamite* (1353) *kṛtam* (*Dṛggaṇita*, Ch. 2, v. 46).
3. *śake akṣaṣaṭtricandra* (1365) *mite parameśvaranāmneyaṃ vadanabhuvāgoladīpikā racitā* (*Goladīpikā* II, Ch. 4, v. 91-92).
4. Vide. S. Sambasive Sastri, *Introduction to Āryabhaṭīya* with the *bhāṣya* of Nīlakaṇṭha (TSS 101); Ulloor Parameśvara Aiyer, *Keralisāhitya Caritra, 2*, p. 117; K.V. Sarma 'Gargya Kerala Nīlakaṇṭha Somayājin', *Journal of the Oriental Research Institute Madras, 26*, p. 24-39, 1958.
5. *Āryabhaṭīya* with *Bhāṣya* of Nīlakaṇṭha Somasutvan. Parts I & II (*Gaṇitapāda* and *Kālakṛiyā*), ed. by K. Sambhasiva Sastri, Trivandrum Sanskrit Series (TSS) nos. 101 & 110, 1931-32. Part III (*Golapāda*), ed. by Suranad Kunjan Pillai, Trivendrum Sanskrit Series No. 185, 1957.
6. The *Tantrasaṃgraha* with commentary *Laghu-vivṛti* by Saṅkara Vārior. Published by Suranad Kunjan Pillai, Trivendrum Sanskrit Series No. 188, Trivendrum, 1958.

was a Bhaṭṭa and belonged to Gārgagotra and the Āśvalāyana school. His favourite deity was Lord Śiva. His father was Jātadeva and had a younger brother named Śaṃkara. He was one of the pupil of Parameśvara's son, Dāmodara.[1] From him, he learnt mainly mathematics and astronomy. Nīlakaṇṭha often refers to Parameśvara as his *paramaguru* and considers him as a great authority. Parameśvara, the great astronomer was still alive when Nīlakaṇṭha went to the house of his teacher Dāmodara; and the young Nīlakaṇṭha might have possibly received some instructions from him.[2]

The scholars[3] are more or less unanimous in assigning Nīlakaṇṭha's time in the period 1443 A.D. to 1543 A.D. This is supported by the fact that he was a pupil of Dāmodara, son of Parameśvara.

Another work *Tantrasaṃgraha*, possibly written by him, which is still in manuscript form, deals with mathematical series, problems of algebra and

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1. *iti kuṇḍagrāmajena gārgyagotreṇa āśvalāyanena bhāṭṭena keralasadgrāmagṛhasthena śrīschetāranya nāthaparameśvarakarunādhikaraṇabhūtavigrahena jātavedaḥputreṇa śaṃkarāgrajena jātavedomātulena dṛggaṇita nirmapakaparameśvara putraśrīdāmodarāttajyotiṣāmoyanena ravita āttavedāntaśāstreṇa subrahmaṇyasahṛdayena nīlakaṇṭhena somasutāviracitavividhagaṇitagranthena iṣṭa vāhupapattinā.........*
   (*Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, Gaṇitapāda Colophon, TSS. 101, p. 180).
2. *tad eva paramācāryo mamāha parameśvaraḥ* /
   (*Āryabhaṭīyabhāṣya*, Golapāda v. 48); also compare with : *ata evoktam asmadācāryena goladīpikāyām* (*Goladīpikā*, Ch. 3, v. 35).
3. Vide. *Journal of the Bombay Branch of the Royal Asiatic Society, 20*, (n.s.) p. 75, 1944.
4. Vide. Mss. of *Tantrasaṃgraha* in the Trippunittura Sanskrit College Library and Adyar Library. C/O. Whish's article,

geometry.[4] Nīlakaṇṭha is the author of many other astronomical works namely, *Golasāra*, *Candracchāyā-gaṇita*, *Candracchāyāgaṇitaṭīkā* etc.[1]

JÑĀNARĀJA (c. 1503 A.D.)

Jñānarāja, the author of *Siddhātasundara*[2] and *siddhāntasundarabīja*[3] was the son of Naganātha. Both the works still remain unpublished. The *Siddhāntasundara* in the Sphuṭādhyāya chapter contains sine table and trigonometrical relations.[4] This was written in 1503 A.D.[5] His *siddhāntasundarbīja* is a separate work on algebra which was written as a sequel to Bhāskara II's *Bījagaṇita*. This *Bījagaṇita* has been repeatedly cited by his son Sūryadāsa.

CITRABHĀNU (c. 1475 to 1550 A.D.)

Citrabhānu, one of the student of Gārgya Nīlakaṇṭha[6], is the author of *Karaṇāmṛta*[7], a work on mathematics in four chapters. It deals with mathematical calculations following the *dṛggaṇita* system. Śaṃkara Vāriyar, the author of *Kriyākramakarī* commentary on the *Līlāvatī* was a student of Citrabhānu.

Transactions of the Royal Asiatic Society of Great Britain and Ireland, *3*, 509-23, 1835; K. Balagangadharan's article, *Journal of the Bombay Branch of the Royal Asiatic Society*, New Series, *20*, 77-82, 1944.

1. Vide our *Bibliography*, p. 155-57.
2. For Mss. vide our *Bibliography*, p. 93-94.
3. One good copy is deposited in the *Sans. College Library, Benares*. Vide Catalogue of the Library, printed by the order of the Government, Allahabad, Vol. 2, p. 257.
4. Vide. p. 17-18 of Ms. deposited in *Asiatic Society Bengal* (No. 8210).
5. Vide. Sudhakara's *Gaṇakataraṅginī*. Entry on Jñānarāja.
6. Beginning verse of the *Karaṇāmṛta*.
7. Mss. at the *Curator's Office Library*, Trivandrum (TC), 663A and B.

Saṃkara himself admitted that he learnt the methods of proof from his teacher, Citrabhānu.[1]

The date of composition of his work is given as 1530 A.D. in the commentary on his work.[2]

ŚAMKARA VĀRIYAR OF TṚKKUṬAVELI (c. 1500-60 A.D.)

Śaṃkara Vāriyar was a student of Nīlakaṇṭha[3] and Citrabhānu[4] He wrote the *Laghuvivṛti*[5] comentary on the *Tantrasaṃgraha* in 1556 A.D. He wrote another work, the *Kriyākramakarī*,[6] an elaborate commentary on the *Līlāvatī* of Bhāskara II—giving the rationale and proof of the theorems and formulae. The work is more or less identical with *Karmapradīpaka* of Nārāyaṇa,[7] only difference is that Śaṃkara's work is more elaborate. The introductory verses of of both the works are same.[8]

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1. *ekaviṃśatidhā kāryamityupadiṣṭaṃ citrabhānunāmnā gaṇitagolayuktividagresareṇa bhūsurottamena / tatra diṅmātramasmābhistad-upadeśavaśādiha likhyate /*
(C/O. K. Kunjunni Rāja's paper, *Adyar Library Bulletin*, *27*, p. 154, 1963). "The methods of twenty-one types have been advised by Citrabhānu the pioneer of the scholars who are well versed in *Gaṇita*, *Gola* and *Yukti*. A few of these are being written in brief by me as an advice."
2. *buddhyonmathyoddhṛtaṃ yat* (1691513 days i.e. c. 1530 years).
3. Vide. beginning of the *Karaṇāmṛta* of Citrabhānu.
4. See. under Citrabhānu, fn. 3.
5. *Trivandrum Sans. Series* (TSS). 188.
6. Oriental Library, Madras, Ms. No. R2754; vide also *Bulletin of the National institute of Sciences of India*, *21*, p. 320, 1963.
7. Ulloor S. Paramesvara Aiyer, *Kerala Sāhitya Caritram*, *2*, p. 121. For. Mss. of *Karmapradīpaka*, vide our *Bibliography*, p. 150.
8. *nārāyaṇaṃ jagadanugrahajāgarūkaṃ*
śrīnīlakaṇṭhamapi sarvavidaṃ praṇamya /
"I bow to Nārāyaṇa, a great benefactor of the world and Śrī Nīlakaṇṭha, a scholar well versed in all śāstras".

JYEṢṬHADEVA (c. 1500 to 1600 A.D.)

Jyeṣṭhadeva[1] is another South Indian scholar on mathematics and astronomy. His *Yuktibhāṣā*[2] is a Malayalam work on mathematics and astronomy. The work is divided into two parts, the first comprising arithmetic, geometry and algebra and the second exclusively devoted to astronomy and allied subjects and spherical trigonometry. The contribution of this unique work is that it gives the rational or derivation of all theorems and formulae then in use among astronomers. The work is mainly based on the *Tantrasaṃgraha* of Nīlakaṇṭha.

A Sanskrit work *Gaṇita Yuktibhāṣā* is available at present, which seems to agree closely with the Malayalam version of the *Yuktibhāṣā*. It is somewhat difficult to decide which is the earlier version and which is the translation.

Jyeṣṭhadeva was a Nambūtiri Brāhman of Paṛaṅṅoṭṭu family in Ālatture village. He was one of the students of Dāmodara. Jyeṣṭhadeva was the teacher of Acyuta Piṣāraṭi of Tṛkkaṇṭiyūr who refers to him in his *Uparāgakriyākrama* (vide infra).

On the date of *Yuktibhāṣa*, there remains some controversy.

Whish[3] and the editors of the published *Yuktibhāṣā* suggested on the basis of a verse available at the end

1. Vide Sarma, K.V. "Jyeṣṭhadeva and his identification as the author of *Yuktibhāṣā*", *Adyar Library Bulletin, 22,* p. 35-40, 1958; Also see. *27,* p. 156-58, 1963.
2. Ed. Ramvarma Maru Thampuran and Akhileswara Aiyer, Trichur, 1947.
3. *Transactions of the Royal Asiatic Society of Great Britain and Ireland, 3,* p. 509-23, 1835.

of a manuscript that the author of the work was Brahmadatta and the date of the work was 1750 A.D. Sarma[1] has shown that the passage contains a word *alekhi* which does not mean 'composed' but 'transcribed' or 'copied' Hence Brahmadatta is a transcriber and the time of his doing so was 1750 A.D.

A manuscript (No. 755) named *Gaṇitayuktibhāṣā* of the Kerala University Mss. Library, Trivandrum contains a fragmentary passage dealing with precision of equinoxes (*ayanacalana*) where a verse (beginning on folio 34 b) refers Jyeṣṭhadeva as the author of *Yuktibhāṣā*[2] The mention of Jeṣṭhadeva is more explicitly mentioned in an old palm-leaf manuscript (No. 9886) of the Oriental Institute, Baroda. It is a Malayalam commentary on the *Sūryasiddhānta* which contains also a chronological list of Kerala astronomers. This list[3] contains the following names :

Parameśvara (1360-1455 A.D.)–son, Dāmodara–pupils, Nīlakaṇṭha Somasutvan (1443-1543 A.D.) and Jyeṣṭhadeva—pupil, Acyta Piṣāraṭi (1550-1621 A.D.)—pupil, Nārāyaṇa Bhaṭṭātiri (1587 A.D.).

Furthermore, in a Malayalam commentary[4], the

1. *Adyar Library Bulletin, 22,* p. 35-40, 1958.
2. *jukriyādike pāte svarṇaṃ tatsādhane vidhau |*
   ityuktā kṣepacalanasyānītis 'tantrasaṃgrahe'
   jyeṣṭhadevo'pi 'bhāṣāyāṃ nādhikaṃ kiṃciduktavān ||
   The *Tantrasaṃgraha* referred to here is the well-known work of that name by Nīlakaṇṭha Somayājin and the *Bhāṣā* referred to in continuation is obviously the *Yuktibhāṣā*, for it is expressly stated in the beginning of the *Yuktibhāṣā* that it was composed as an exposition of the *Tantrasaṃgraha.*
3. Vide also *Adyar Library Bulletin, 22,* p. 35-40, 1958.
4. Ulloor S. Parameśvara Aiyer, *Kerala Sāhitya Caritram, 2,* p. 323.

date of completion of the *Yuktibhāṣā* in number chronogram of the *kaṭapayādi* system, is given as 1, 714, 262 corresponding to 1592 A.D. This shows that Jyeṣṭhadeva must have flourished during the later part of the sixteenth century.

KARAṆAPADDHATI (1596 A.D)

The *Karṇapaddhati* is an important astronomical work in ten chapters by an unknown Kerala astronomer of uncertain date. Only this much is known about the author that he was a Brāhmin who took his abode in the village Śivapura.[1] The text in Devanāgarī script is published by K. Sambaśiva Śāstrī[2], but this edition misses the opening verse.[3] Apart from the usual elements and formulae characteristic of Hindu astronomy, the work gives in the sixth chapter, trigonometrical π, *sine cosine* and *tan* series. The mathematical importance of not only *Karaṇapaddhati* but three other texts namely *Tantrasaṃgraha, Yuktibhāṣā* and *Sadratnamālā* was first pointed out by C.M. Whish[4] who discussed π and tan functions given in the texts and attempted to fix their dates.

1. The last verse of the tenth chapter reads : *iti śivapuranāma grāmajaḥ ko'pi yajvā.* The author has been identified as Putumana Samayaji, vide *All India 18th Oriental Conference Bulletin*, p. 562-64.
2. *Trivandrum Sanskrit Series* (TSS) No. 126.
3. Vide *Madras Triennial Catalogue*, Vol. II, Part I, Sans. A. Call No. *1310*.
4. 'On the Hindu quadrature and the infinite series of the proportion of the circumference to the diameter exhibited in the four śāstras : *Tantrasaṃgraha. Yuktibhāṣā, Karaṇapaddhati* and *Sadratnamālā*', *Transaction of the Royal Asiatic Society* (TRAS) of Great Britain and Ireland, *3*, part III, 509-523, 1835.

As regarding the date of *Karaṇapaddhati*, there is some controversy.[1] Recently it is traced that the verse 11 of the chapter 10 of the *Karaṇapaddhati* contains sufficeint hints as to the date of writing of the text in 1596 A.D. The number 554754 (*viṣṇosuvṛśamaṇu*) when lessened by 596 (*taddhāma*), gives the date of the text in Śaka days i.e., 554158 Śaka days=1518 Śaka year=1596 A.D.

ACYUTA PIṢĀRAṬI (c. 1550 to 1621 A.D.)

Acyuta Piṣāraṭi, a non-Brāhmin astronomer mathematician of Kerala was a student of Jyeṣṭhadeva. His *Karaṇottama*[2] is a manual of mathematics in five chapters containing 109 verses. In a verse[3] of his astronomical work, *Uparāgakriyākrama*,[4] he expressed that at the time of composition of this work, Jyeṣṭhadeva had become very old, but was still alive. He belonged to Piṣāraṭi community and referred to as Vaiṣṇava whose function was to look after the external affairs of the temple. His pupil, Nārāyaṇabhaṭṭa gives the date of his death as 1621 A.D. from the study of a chronogram.[5]

1. Vide, Bag, A. K. 'Trigonometrical Series in the Karaṇapaddhati and the probable date of the text', *Indian Journal of History of Science*, *1*, No. 2, p. 98-106, 1966.

2. Vide Mss. at the *Curator's Office Library, Trivandrum* (TC) 697B.

3. *prokta pravayaso dhyānājjeṣṭhadevasya sadguroḥ vicyutāśayadośeṇetyacyutena kriyākramaḥ* ||

   "The *Kriyākrama* is being written by Acyuta to modify the faults of analogies of the work of my beloved aged guru Jyeṣṭhadeva".

4. TC. 655B.

5. K. Kunjunni Rāja, *Adyar Library Bulletin*, *27*, p. 160, 1963.

NĀRĀYAṆA (c. 1588 A D.)

Nārāyaṇa was the son of Govinda and teacher of Muniśvara.[1] The manuscript of his *Khaṇḍitabījagaṇita* or shortly *Bīja*, a work on algebra is lying in Sanskrit College Library, Banaras.[2] He is different from his predecessor Nārāyaṇa, the author of *Gaṇitakaumudī*. All the sūtras of the *Bīja* are composed in *āryā* metre. The Ms. is incomplete at the end. His time is still not known, but he can be tentatively fixed 15 years earlier than that of Muniśvara (e. 1603 A.D.), since Muniśvara himself admitted that Nārāyaṇa was his *guru*.

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1. *Geneology*

Rāma (Dadhigrāma, Devarāta gotra)
- Trimalla
  - Vallāla
    - Rāma
    - Kṛṣṇa (Bījāṅkurakāra)
    - Govinda
      - Nārāyaṇa (c. 1588 A. D.)
- Gopirāja
  - Raṅganātha
    - Muniśvara (c. 1603 A. D.)
  - Mahādeva

2. For Ms. reference vide our *Bibliography*, p. 150.

MUNĪŚVARA (1603 A.D.)

Munīśvara, son of Raṅganātha, was the main astronomer in the Court of King Sājāhān. His parents belonged to Devarāta gotra and lived in the Dadhigrāma of Alicpur Province near the river Payaṣṇi. He was born in 1603 A.D.[1] He wrote *Niṣṛṣṭārthaduti*, a commentary on the *Līlāvatī* of Bhāskara II and *Pāṭīsāra*, another work on mathematics besides several works on astronomy[2]. Both the works of mathematics still remain unpublished. His *Siddhāntasārvabhauma* and his own commentary on the *Siddhāntasārvabhauma* were written in 1646 A. D. and 1650 A. D. respectively[3]. Munīśvara was called by another name, Viśvarūpa. He had a controversy with Kamalākara and tried to refute some of the latter's contribution.

KAMALĀKARA (c. 1616 to 1700 A.D.)

Kamalākara, the son of Nṛsiṃha Daivajña was the younger brother as well as disciple of Divākara Daivajña. He was a Mahārāṣṭrīya Brāhmaṇa. He composed his work, *Siddhāntatattvaviveka*[4] in 1580

1. Vide. Preface of the *Siddhāntasārvabhauma*, Princess of Wales Saraswati Bhavan Text no. 41 (Part I).
2. Vide. Sudhākara's *Gaṇakataraṅginī*—Munīśvara; For other works, vide our *Bibliography*, p. 145-46.
3. Vide. Sudhākara's *Gaṇakataraṅginī;* also Preface of the *Siddhāntasārvabhauma* mentioned above.
4. Ed. with notes by Sudhākara Dvivedi and Muralidhara Jhā, Benares Sanskrit Series, Benares, 1885; Ed. with a comm. by Gaṅgādhara Miśra Śarmā, Lucknow, 1929, etc.

Śaka era i. e., 1658 A.D. He was a follower of *Sūrya-siddhānta* and tried to refute some of the views of Bhāskara II and Munīśvara. Though *Siddhāntatattva-viveka* is mainly a work on astronomy, he made contribution to trigonometry by giving several correct relations between chords and its corresponding arcs in his *jyotpatti* section of Spaṣṭādhyāya. His other works are : *Grahagolatattva*, *Grahasāraṇī*, *Kairāśyudāharaṇa*, a commentary on the Bhāskarācārya's *Līlāvatī*, *Manoramā*, a commentary on the *Grahalāghava* of Gaṇeśa Daivajña, *Sauravāsanā*, a commentary on the *Sūryasiddhānta*, and *Śeṣavāsanā*, a supplement to the author's *Siddhāntatattvaviveka*[1]

In the concluding verse of his *Siddhāntatattvaviveka*, he admitted that his ancestors lived in the village Golagrāma near the river Godāvarī.

PAṆḌIT JAGANNĀTH (c. 1657 to 1750 A.D.)

Paṇḍit Jagannāth worked under the patronage of king, Jayasiṃha, who was well known for his interest in the study of astronomy and mathematics. He came to Jayasiṃha at the age of twenty. He was a scholar in Sanskrit and acquired a great proficiency in both Persian and Arabic languages in a very short time[2]. He translated Euclid's *Elements* from an Arabian version, *Taḥrīr-u-Uqlidas* by Nasīra'd-Dīn aṭ-ṭūsī (1201 A.D.) under the name *Rekhāgaṇita*[3].

1. For details of editions vide p. 49.
2. vide *Rekhāgaṇita*, edited by Kamalāśaṅkara; notes on Jagannāth in the Preface.
3. The *Rekhāgaṇita*, two volumes. Ed. with a critical preface, introduction and notes in English by Kamalāśaṅkara Trivedi, *Bombay Sanskrit Series* nos. 61-62, Bombay 1901-2.

His other works are : *Siddhāntasamrāṭ*[1], a Sanskrit version of Ptolemy's *Almagest*, and *Siddhāntasārvabhauma* containing partly *Samrāṭsiddhānta* and partly Hindu astronomy.

ŚAMKARAVARMAN (c. 1800-38 A.D.)

Śaṃkarvarman, the author of *Sadratnamālā*[2] flourished in the first half of the nineteenth century A.D. He was born in 1800 A.D. He wrote his *Sadratnamālā* under the patronage of Rāmavarman, an heir-apparent of King Udayavarman in North Kerala[3]. The *Sadratnamālā* is a short work in five chapters and gives various results of the Kerala mathematicians without deductions. The date of composition of the work has been determined from the chronogram : *lokāmbāsiddhasevye* i.e., 1,797,313 corresponding to 1823 A.D. He wrote also a Malayalam commentary on the work. He died in 1838 A.D.

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1. Edited by Ramswarup Sarma, Indian Institute of Astronomical and Sanskrit Research, 3 vols, New Delhi, 1967.
2. Published in Malayalam script with the commentary of K. Kunjunni Rāja. Also see Adyar Library Ms. 67735 and Madras Oriental Mss. Library, Ms. No. R. 4448.
3. Introductory verse; also see. *Journal of the Bombay Branch of the Royal Asiatic Society, 20*, p. 74 fn.

CHAPTER II

ARITHMETIC

The necessity for the development of a systematic method of counting in an organised social life led to the birth of arithmetic in all countries at the dawn of civilization. The Babylonians developed a sexagesimal system of expressing numbers in cuneiform writing on clay tablets. The Egyptians developed a system of numeration, using hieroglyphic notation besides hieratic and demotic ones. Fractions were a subject of great difficulty to the Egyptians as well as to the Babylonians, Greeks and Romans. Indians were responsible for the introduction of the decimal place-value system and more or less modern form of numerals with the symbol for zero, for the development of simplified and straight forward methods of fundamental operations, and for the calculation by rule of three. The Indian numerals with zero, the principle of position (decimal place-value) and some of the fundamental operations found their way to Arabia by the efforts of Abbāsid Khalifs (8th to 10th century A.D.) at Bagdad.

Numbers

The Sanskrit names for the counting of numbers used in Vedic India from one to nine are : *eka*, *dvi*, *tri*, *catur*, *pañca*, *ṣaṭ*, *sapta*, *aṣṭa*, *nava*. Besides 10, multiples of 10 upto 100, separate names for numbers were recognised and named as follows : *daśa* (10), *viṃśati* (2×10), *triṃśat* (3×10)...etc. For larger numbers, ten and multiples of ten had formed the basis of numeration. While the Greeks had no terminology for denomination above myriad (10^4) and the Romans above the mill (10^3), the ancient Indians dealt freely with upto twelve denominations. The *Yajurveda Saṃhitā* thus gives : *eka* (1), *daśa* (10), *śata* (10^2), *sahasra* (10^3), *ayuta* (10^4), *niyuta* (10^5), *prayuta* (10^6), *arbuda* (10^7), *nyarbuda* (10^8), *samudra* (10^9), *madhya* (10^{10}), *anta* (10^{11}) and *parārdha* (10^{12}). The same list occurs in the *Taittirīya Saṃhitā*, *Maitrāyaṇī Saṃhitā* and others.[1] For expressing compound numbers lying between 10 and multiples of 10 upto 100, addition of simple numbers (from 1 to 9) was made to the required multiples. For example, *ekādaśa* ($=1+10=11$) *saptaviṃśati* ($=7+2\times 10=27$), *aṣṭatriṃśat* ($=8+3\times 10=38$) etc. Compound numbers above 100 were expressed as follows : *ṣaṣṭiṃ sahasrā sapta śatāni navatiṃ nava* ($=60\times 10^3+7\times 10^2+9\times 10+9=60799$). In certain special cases, the principle of subtraction was also in evidence. Thus number 19, 29 etc. were expressed as *ekānnaviṃśati* ($=20-1=19$), *ekānna triṃsat* ($30-1=29$) respectively. In the *Sūtra* period, *ekānna* was changed to *ekona*. The principles explained above were also followed in the *Śulba-sūtras* and later works. In one instance, 972 is expressed as *aṣṭaviṃśatyūnaṃ sahasram*$=1000-28$ (*Āpastamba-śulba*, 5. 7). The Babylonians (c. 1600 B. C.) introduced earlier sexagesimal scale of notation.[2] There is no evidence that the decimal

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1. *Yajurveda Saṃhitā*, 17. 2; *Taittirīya Saṃhitā*, 4. 40. 11. 4 and 7. 2. 20. 1; *Maitrāyanī Saṃhitā*, 2. 8. 14; *Kāṭhaka Saṃhitā*, 17. 10; vide also *History of Hindu Mathematics* by Datta and Singh, Part I, Motilal Banarasi Das, Lahore, p. 9, 1935.

2. Cajori, F. *A History of Mathematics*, 2nd revised and enlarged edition, p. 4.

scale of the Indians was derived from the Babylonians, though the principle might be the same. The decimal scale, as it could be seen afterwards, is in fact far more convenient and simple and is a natural way of expressing numerical value which has been adopted subsequently all over the world as the ideal system.

Classification of Numbers

Classification of numbers into odd and even first appeared in the Vedic literature.[1] The classification also appeared in the Jaina works but the authors did not consider unity a number like the ancient Greeks. In the Jaina work, *Anuyogadvāra-sūtra*[2], the numbers were classified also as numerable (*saṃkhyeya*), innumerable (*asaṃkhyeya*) and infinite (*ananta*). The highest numerable number of the Jaina corresponds to Alef-zero[3] of the modern mathematics and they also made an attempt to define number beyond Alef-zero. As regards innumerable number, names of several sub-classifications were mentioned but the idea was not very clear. Idea of infinity was expressed in connection with different dimensions namely infinity in one, two, three and infinite dimensions.

Indices

For representation of large numbers, different specifications seem to have been necessitated to indicate them correctly. For instance, the raising of a

1. *Taittirīya Saṃhitā*, 7. 2. 12-13.
2. Datta, B. 'The Jaina School of Mathematics', *Bulletin of the Calcutta Math. Society, 21*, p. 136-38, 1929.
3. The cardinal number of the aggregates of all the finite integers 1, 2, 3, ... n is called Alef-Zero. The theory of such numbers was fully developed by G. Cantor in 1883 A. D.

number to its own power was technically known as *varga saṃvargita*. First *varga saṃvargita* of $a = a^a = b$, say; second *varga saṃvargita* of $a = b^b = (a^a)^{a^a} = c$, say. Third *varga saṃvargita* of $a = c^c$ etc. In this process third *varga saṃvargita* of 2 comes out to be $(256)^{256}$, a number higher than the number of electrons in the universe. To express a large number, Jaina authors (c. 300 B.C.) employed powers of integers now known as laws of indices, examples like :

$$a^m \times a^n = a^{m+n}, \ (a^m)^n = a^{mn}$$

are found in their texts.[1]

Logarithms

The expression $\log_2 8 = 3$ means the number 8 can be divided by 2 three times. This was defined by Napier (1550–1617) who has been given priority with regard to the publication of a table of logarithms and an account of their meaning and use. In the earliest centuries of the Christian era Jainas (vide *Dhavalā* commentary of Vīrasenācārya) conceived the idea of logarithms to the base 2, 3, 4, though no general use of the idea seems to have been made by them. They used *ardhaccheda*, *tṛkaccheda*, *caturtha-ccheda* of a number, as the number of times it can be divided by 2, 3, 4, etc. For instance *ardhaccheda* of $x = \log_2 x$, *tṛkaccheda* of $x = \log_3 x$ etc. Moreover, there is evidence in the *Dhavalā*[2] commentary that the following rules were known to them.

1. *Bulletin of the Calcutta Mathematical Society*, *21*, p. 136-38, 1929.
2. Singh, A. N. 'History of Mathematics in India from Jaina Sources', *Jaina Antiquary*, *15*, no. 2, 46-53, 1949.

(1) $\log \frac{m}{n} = \log m - \log n$

(2) $\log (m.\ n) = \log m + \log n$

(3) $\log_2 (2)^m = m$

(4) $\log (x^x)^2 = 2x \log x$

(5) $\log \log (x^x)^2 = \log x + 1 + \log x$; the expression log log x is known as *vargaśalākā* of x.

(6) $\log (x^x)^{x^x} = x^x \log x^x$ etc.

Fraction

Few fractions are mentioned in the Vedic literature. *Ardha*, *pāda*, *sapha* and *kalā* denote $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ respectively. But only the first two are common. The word *pāda* was derived from four parts of a stanza; a stanza in Sanskrit literature is divided into four parts from the Vedic times, each of which is called a *pāda*. The idea of a fraction of $\frac{1}{16}$ was derived from the increasing visibility of the moon from the new moon to fullmoon covering the period of sixteen days and thus the moon is said to possess 16 *kalās* or parts. Mention of the number of fractions are also found in the *Ṛgveda*, *Taittirīya Saṃhitā*, *Taittirīya Brāhmaṇa* etc.[1] In the *śulbas*[2], the fraction is denoted by the term *bhāga*, e.g., *pañcamabhāga*=$\frac{1}{5}$, *daśamabhāga*=$\frac{1}{10}$ etc. A number followed by a fraction was denoted in the *śulba* as follows: *ardhacaturtha*

1. Datta, B. and Singh, A. *History of Hindu Mathematics*, Part I, Motilal Banarasi Das, Lahore, p. 185, 1935.
2. Datta, B. *The Science of Śulba*, University of Calcutta, p. 212-16, 1932.

$=3\frac{1}{2}$, *ardhapañcāśat*$=49\frac{1}{2}$ etc. Four fundamental arithmetical operations with elementary fractions were known to the *śulbakāras*. Some expressions other than unit fractions were also used in the *Śulba-sūtras*[1] (c. 600 B.C.), *Arthaśāstra*[2] of Kauṭilya (3rd century B. C.), *Vedāṅgajyotiṣa*[3] (200 B.C.), *Sūrya-prajñapti*[4] (200 B.C.), etc. In the *Śulba-sūtras*, fractions were expressed in language. The *Bakhshālī Ms.* (4th century A.D.)[5] first expressed operations of fraction in terms of symbols both with and without the dividing lines. This is followed in the *Brāhmasphuṭasiddhānta* of Brahmagupta (628 A.D.) and *Gaṇitasārasaṃgraha* of Mahāvīra (c. 850 A.D.) and in later works. For addition and subtraction of fractions, Brahmagupta suggested the operation of reduction to a common denominator. The works of Mahāvīra

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1. *caturtha-saviśeṣa saptama*$=\frac{1}{7}\left(\frac{1}{4}\sqrt{2}\right.$ (*Āpastambaśulba*, 19. 4; 19. 7); *caturdaśa prakramān triṃśca prakrama saptabhāgān* $=14\frac{3}{7}$ *prakramas* (*Kātyāyanaśulba*, 6. 2) etc.
2. Shamasastry, R. *Kauṭilya's Arthaśāstra* (Eng. tr.), fifth ed., p. 113–21, Mysore, 1956.
3. *kalā daśa saviṃśa nāḍikā* $=10\frac{1}{20}$ (*Vedāṅga Jyotiṣa*, v. 38).
4. *do joyanāhaṃ addhaduyātvaliśate sitayabhāge* $=2\frac{42\frac{1}{8}}{183}$ (*Sūryaprajñapti*, sūtra 18).
5. $\begin{matrix}11\\1\end{matrix}$ *yu* $\begin{matrix}5\\1\end{matrix}$ means 11+5; $\begin{matrix}2\\1\\2\end{matrix}$ represents $2\frac{1}{2}$; $\begin{matrix}1\\1\\2\end{matrix}$ as $1\frac{1}{2}$;

$\left|\begin{matrix}1 & 1 & 1 & 1 & bha\\ 1 & 1 & 1 & 1 & \\ 2+ & 3 & 4+ & 5 & \end{matrix}\right|\begin{matrix}36\\1\end{matrix}$ means $\frac{36}{(1-\frac{1}{2})\,(1+\frac{1}{3})\,(1-\frac{1}{4})\,(1+\frac{1}{5})}$

and Śrīdhara contain special rule for their reductions to proper fractions. The fraction was denoted as *bhinna* by Mahāvīra (c. 850 A. D.) and Śrīdhara (c. 903 A.D.). The numbers of the form

$$\left(\frac{a}{b} \pm \frac{c}{d}\right), \quad \left(\frac{a}{b} \text{ of } \frac{c}{d}\right), \quad \left(\frac{a}{b} \pm \frac{c}{d} \text{ of } \frac{a}{b}\right)$$

were termed *jāti* (i.e., classes).

Amongst other nations[1], Babylonians (c. 2000 B.C.) on their cuneiform clay tablets included some special symbols for $\frac{2}{3}$, $\frac{2}{18}$, $\frac{4}{18}$ and $\frac{5}{7}$ etc. Egyptians, Greeks as well as the Romans dealt with unit fraction. Chang Chiu-Chien (6th century A.D.) remarked in the preface of his *Arithmetical classic* that the chinese found it difficult to express a fractional number and to perform any operation with them. Even as late as 10th century A.D. Rabbi Saadia ben Joshep al-Fayyaumi, a Hebrew writer living in Egypt made wide use of the unit fraction in his computations.

Numerals

The early symbolic numerals of the world were of various types. The characters varied to meet the linguistic conditions of the different cultural areas in different parts. Here main types of numerals used by the Ancient Babylonians, Egyptians, Chinese, Greeks, Romans, Indians and Arabians are tabulated below.

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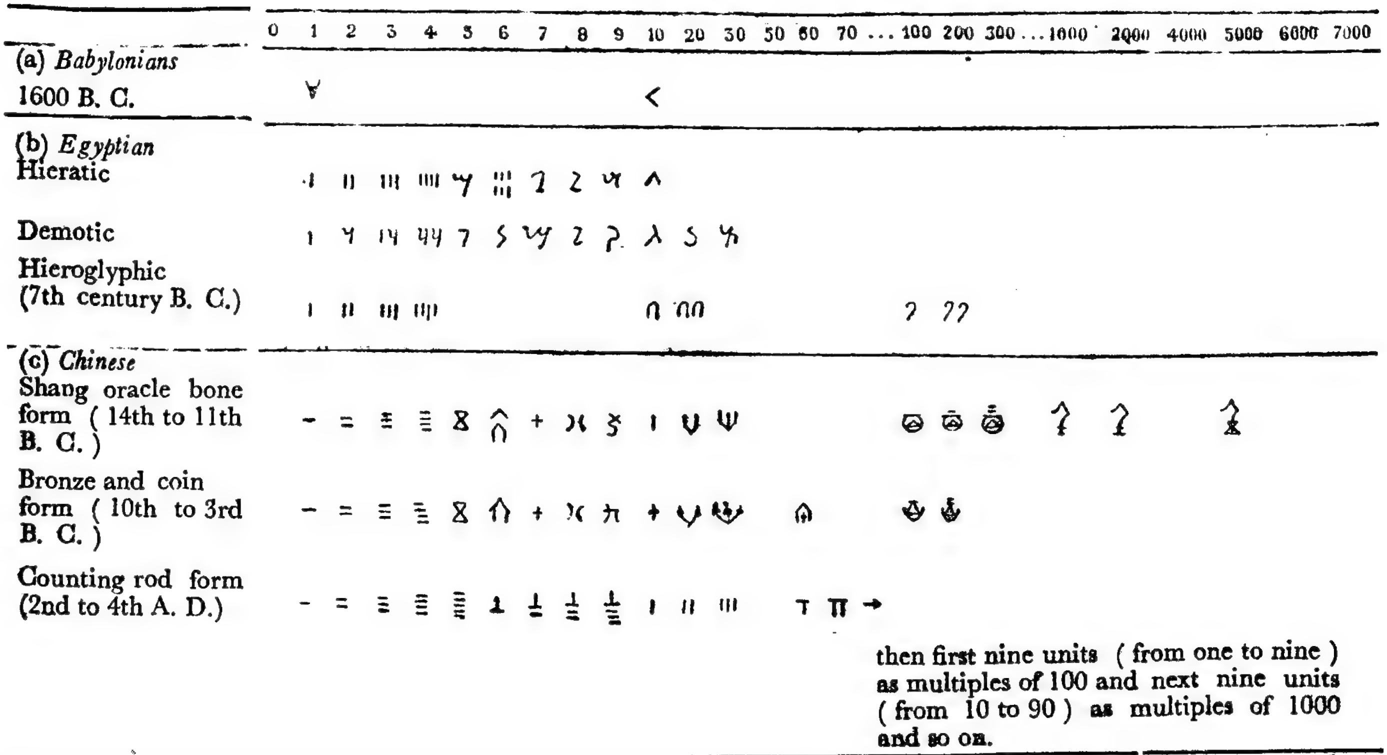
1. Smith, D. E. *History of Mathematics*, 2, Dover Publication, p. 210-215.

TABLE OF NUMERALS



Contd....

TABLE

(Contd.)

	0	1	2	3	4	5	6	7	8	9	10	20	30	50	60	70	...	100	200	300	...	1000	2000	4000	5000	6000	10000
(d) *Greek*																											
Geometric (5th century B. C.) Attice system						Γ					Δ							H				X					
Alphabetic system (300 B. C.)		α	β	γ	δ	ε	ϛ	ζ	η	θ	ι	κ	λ					ρ	σ	τ		,α	,β	,δ			
(e) *Roman*		I	Ⅱ			V					X			L				C				M					
(f) *Indian*																											
Kharoṣṭhī (250 B. C.)		I	II	III	X	IX	IIX		XX		7	3		733				ʎI	ʎII								
Brāhmī-Aśokān (250 B. C.)		I	II		+		6							C				ਮ	⁚								
Nānāghāt (150 B. C.)		–	=	≡	¥		ϥ	7		?	α	o			⊣			ℋ	ℋ.			ℸ					
Nāśik (100 A. D.)		–	=	≡	¥	h	ϥ	7	ソ	?	α	⊟	১					ℷ	ℷ	ℱ		ց	ग				
Bakhshālī Ms. (400 A. D.)	•	^	३	३	४	५	६	७	८	९	→ place value.																
Sanskrit form (10th century A. D.)	०	१	२	३	४	५	६	७	८	९	→ place value.																
(g) *Modern Arabic* (10th century A.D.	٠	١	٢	٣	٤	٥	٦	٧	٨	٩	→ place value.																

(a) *Babylonians :* The Babylonians wrote with stylus on clay tablets the result being wedge-shaped characters. The tablets were then baked in the sun. The numbers were usually expressed with the help of a sexagesimal scale. Practically 59 symbols from one to fifty-nine (made by two basic symbols for one and ten) were used. Whenever a symbol for sixty was required, it was replaced by 1 of the next higher order as follows :

$$1.10 = 1 \times 60 + 10 = 70$$
$$2.12 = 2 \times 60 + 12 = 132$$

In this way a big member was expressed by 1. 41. 33. 59. 3. 45. This obviously involves an idea of place-value. The numerical system is not very happy, because, a comparatively greater amounts of repetition is unavoidable for obtaining a number below fiftynine. In many cases, the numerical value of a symbol had to be determined from the context.[1] For example, in certain tablet 71 is represented by symbols of 1.10.1 i.e. $1 \times 60 + 10 + 1 = 71$. The number $171\frac{1}{2}$ was represented by symbols of 2.50. 1. $\frac{1}{2}$ i. e. $2 \times 60 + 50 + 1 + \frac{1}{2} = 171\frac{1}{2}$. The symbol for 1.12 may be expressed as 1.12.0=4320 or as 1.12=72 or as 1; $12 = 1\frac{1}{5}$ etc. written sexagesimally.

(b) *Egyptians and the Semitic Groups :* The Egyptians used stone, papyrus, wood and pieces of pottery for writing. They in their Hieroglyphic system of writing possessed separate symbols for one, ten, hundred, thousand, ten thousand, hundred thousand and million. The Hieratic (religious) and Demotic (popular) system of writings had signs of symbols

1. Smith, D. E. *Ibid.*, 2, p. 37-38; Neugebauer, O. *The Exact Sciences in Antiquity*, p. 20, 26, Copenhagen, 1951.

for one, five to ten, twenty, thirty,......to hundred. The Hieroglyphic and Hieratic numbers were commonly written from right to left and also from left to right and sometimes as in early inscriptions from top downwards. The numbers were expressed by repetition and addition of these symbols as necessary. As for example, the number 23 was represented by IIIUU i.e. $1+1+1+10+10=23$; this was also written as UUIII. They therefore had little idea of place value.

The Phoenician use symbols for one, ten, twenty and hundred; the Palmyrene for one, ten, twenty and hundred. There is no evidence that they had any idea of place value.

(c) *Chinese* : Of the several systems used by the Chinese, those of Shang oracle bone form (14th to 11th century B.C.), Bronze and Coin form (10th to 3rd century B.C.) and the Counting rod forms (2nd century B.C. to 4th century A.D.) deserve special notice in this connection.[1] Their symbolic numerals were written usually from top downwards with their values arranged in the descending order.

In the Shang oracle bone form, separate symbols for one, five, six, seven, eight, nine, ten, twenty, hundred, thousand etc. were used. For two to four, the symbol for one was repeated and other numbers were obtained by similar repetitions. For instance, 162 was expressed by the symbols of 100, 60 and 2 placed in the left to right order.

In the Bronze and Coin form (10th century B. C. to 3rd century B. C.) and other forms found on

1. Needham, J. *The Science and Civilisation in China, 3*, p. 6-7, Tables 22 and 23, Cambridge, 1957.

coins of Chou period (6th to 3rd century B.C.) more or less similar symbols as those of Shang oracle bone were employed. Neither improvement of method nor use of place-value was known to them.

The Chinese in their Counting rod forms between 2nd century B.C. to 4th century A.D. used separate symbols for one, six, ten, sixty. Symbols for one was repeated to express numbers upto five and the values for seven to nine were represented by successive addition of symbols for one to that of six. For the numbers of the next higher order than 100, different symbols were used.

(d) *Greeks* : The Greeks initially were deeply interested in numbers which were connected with geometric forms.[1] They had two other systems of notation—the Attice system and the Alphabetic system. The former, made of upright strokes and initials of several number names, was no better than the systems prevalent among the Egyptians and other contemporary civilisations. This system was replaced at about third century B.C. by the alphabetic system of numerals which were commonly known among the Hebrews, the Syrians, the Arabs, the Persians and other Semitic people. The alphabetic system had twenty-four letters in their common Ionic alphabet, to which they added three Phoenician alphabets to make it twenty-seven. These are divided into three groups, first nine alphabets denoting nine units, second nine alphabets denoting nine multiples of ten and the third nine alphabets denoting nine multiples of hundred. They wrote from left to right. In wri-

1. Smith, D. E. *History of Mathematics*, 2, p. 24-25, Dover Publication.

ting a compound number, the element of the higher denominations was put first. For examble, 43, 678 was represented by the symbols of four myriad (one myriad=ten thousands), three thousand, six hundred, seventy, and eight maintaining an additive principle.

(e) *Romans* : The Roman numbers were expressed in terms of symbols for one, five, ten, fifty, hundred, five hundreds, and thousand. The normal mode of writing was to put the greater elements in the beginning. For example, the number 1548 was written as MCCCCCXXXXVIII i.e. 1000+500+40+8. In certain cases, it was reversed to denote subtraction. But on a longer view, i.e. in a compound number of still greater value, it will be found that the normal mode has been preserved on the whole. The system had the advantage of using only few symbols for expressing any number in a very simple way. This is the reason why it received a considerable popularity and is still adopted occasionally in numbering the chapter and year of books.

(f) *Indian* : It has already been discussed how the word numerals on a decimal scale were used by the Vedic Indians to express numbers. Regarding notational use, the earliest example of the use of symbols for the expression of numbers in India is provided by Kharoṣṭhī (250 B.C.) and Brāhmī (250 B. C.) inscriptions.[1]

1. Bhau Daji, 'The Ancient Sanscrit Inscriptions in the cave inscriptions', *Journal of the Bombay Branch of the Royal Asiatic Society, 8*, p. 231, 1868; Smith, D. E. *History of Mathematics*, 2, p. 66, Dover Edition; Das, Sukumar Ranjan. 'The Origin and Development of Numerals', *Indian Historical Quarterly, 3*, p. 97-120, 356-57, Calcutta, 1927.

The Kharoṣṭhī was particularly script of North-Western India (including Afganisthan and the Northern Punjab). It disappeared in the 3rd century A.D. In the Kharoṣṭhī numerals, there were separate symbols for one, four, ten, twenty and hundred. They were written from right to left with lower elements before the higher. For example, the number 122 was expressed by symbols of 2, 20, 100, and 274 by 4, 10, 20, 20, 20, 200. But in cases of the hundred and its multiples, the symbols for the smaller elements were placed after the symbol for hundred to denote multiplication. All these were common with the principles underlying other systems of Semitic numeral notation which we have discussed before.

The Brāhmī numeral system consisted of separate signs for one and four to nine, two and three were denoted by repetitions of the symbols for one. It had also separate symbols for ten and its multiples as well as four hundreds upto seventy thousands. The plan was indeed nearly the same, as those of the Demotic, Hieratic and the Chinese. The symbols were written from left to right beginning with symbols of higher values. For example, 274 was written with the help of the symbols for 200, 70 and 4. The Nānāghāt and the Nāśik cave inscriptions amongst few other inscriptions namely Kuṣāṇa (150 A.D.), Kṣatrapa (200 A.D.) and Gupta (c. 300-450 A.D.) are noteworthy. Both these types (vide Table) bear a close relation and resemble each other. According to many foreign scholars, both Nānāghāt (150 B.C.) and the Nāśik (100 A.D.) types of numerals form the basis of our third or the modern numeral system.

This third numeral system, (given in the *Bakhshālī MS.*—vide Table) which contains only ten symbols from one to nine and the symbol for zero could express any number however large on decimal place-value system. In this system the values of the notational places, *eka*, *daśa*, *śata* etc. increase either from right to left or from left to right. The numbers are always read from higher to lower order. This system is identical with the system which as we have seen appeared later amongst the Arabs during the tenth century of the Christian era. From there it went to Europe. And this system has now been adopted by all the civilized peoples of the world.

(g) *Arabic* : Bīrūnī (973 1048 A. D.) has referred to two types of notation of numbers namely, the alphabetic (*abjd*) system as *Hurūf al-jummal* or *Hiṣāb al-jummal* and the modern numerals as *al-arqam al-hind* (Indian numerals). Numerous applications of alphabetic numerals are also available in his *Chronology of the Ancient Nations.*[1] Bīrūnī in his *Kitāb al-tafhim li-awā'il sinā'at at-tanjīm* (The Book of Instructions in the Art of Astrology) has given both these notation of numbers. The alphabetic system *(Hurūf al-jummal)*, as used by Bīrūnī was apportioned in orderly sequence to 26 alphabets though initially there was no particular order in their use. It was based on the values of the alphabets without use of decimal place order. This notation seems to have been used exclusively by Arabian astronomers. The other system (*al-arqam al-hind*) given by Bīrūnī and others is shown in the table (vide p. 60). In the tenth and a few

1. Sachau, E. C. *The Chronology of the Ancient Nation*, London, 1879; the work is a English translation ofthe Arabic original.

following centuries it is found that several mathematical works were composed by the Arab scholars having titles containing the words, *al-hind*, *al-hindi*, *al-takht*, *al-ghobar*.[1] These contained methods and systems very closely resembling those of the Indians.

Symbol for Zero and Decimal Place-value System[2]

The concept of 'nothingness' or 'void' which is inherent and instinctive to man of all ages found its expression in the philosophical and literary works of all ancient civilised nations. The feeling of necessity to give a symbolical expression of this concept in mathematical calculation which would permit the representation of a number of only high order without any limit led ultimately to discovery of a symbol for what is now known as zero.

The concept of 'nothingness' or 'void' in mathematical system was first introduced by the Babylonians (14th century B.C.) who left an empty space in between two other numeral symbols along with their sexagesimal place-value system.[3] But this gave rise to much confusion for giving any quantitative expression of the numerical value involved as there can be no fixed standard for an empty space

1. Datta, B. 'Testimony of Foreign Scholars on the Origin of the Decimal Place-value Numerals', *Bulletin of the Calcutta Mathematical Society*, *24*, p. 193, 1932; Saidan, A. S. 'The Developement of Hindu Arabic Arithmetic', *Islamic Culture*, *39*, p. 209-21, 1965.
2. Bag, A. K. 'Symbol for Zero in Mathematical notation in India', *Boletin de la Academia Nacional de Ciencias*, Tome 48, Primer Congreso Argentino de Historia de la Ciencia, Primera Parte, p. 274-54, 1970.
3. Neugebauer, O. *The Exact Sciences in Antiquity*, p. 20, Copenhagen, 1951.

to allow for its unambiguous repetitions. It therefore, helped very little in the symbolical expression of mathematical figures.

In certain Greek astronomical Papyri Mss. (100 B.C.) as well as Ptolemy's table of chords (150 A.D.), symbols resembling that of modern zero were used to indicate the ending of a sentence or a marginal gap.[1] According to some scholars, this served as the origin of the introduction of the symbol for zero in mathematical notation. There is no justification of such a view as their application in mathematical calculation could not be traced in any of the ancient contemporary Greek mathematical works or thereafter.

In India, the application of zero as a part of the numeral system, is characterised by its two types of uses as : A) word-numerals, B) symbolic-use.

(A) *Word-numerals* : The word-numerals were based on a decimal scale of notation namely *eka*, *daśa*, *śata*, *sahasra*......upto 12 places from the time of *Saṃhitās* (1500 B.C.).[2] A large number like, *ṣaṣṭhīm sahasrā sapta śatāni navatim nava* (60, 799) was expressed by placing the word numerals of higher order to be followed by one of lower order in the scale. This successive placing of, *sahasra*, *śata*, *daśa*, *eka* for the expression of a large number in left to right order of gradually decreasing values obviously shows that

1. Neugebauer, O. *Ibid.*, p. 10, 13-14.
2. *Yajurveda Saṃhitā*, 17. 2; *Taittirīya Saṃhitā*, 4. 40. 11. 4 and 7. 2. 20. 1; *Maitrāyaṇī Saṃhitā*, 2. 8. 14; *Kāṭhaka Saṃhitā*, 17. 10; vide also *Hist. of Hindu Mathematics* by Datta and Singh, Part I, Motilal Banarasi Das, Lahore, p. 9, 1935.

the idea of decimal place-value was implied.[1] The scale is undoubtedly more convenient than the Babylonian sexagesimal scale. The *Śatapatha Brāhmaṇa*, *Taittirīya Brāhmaṇa*, *Chāndogya Upaniṣad*, *Vedāṅga Jyotiṣa* etc. had similar uses of numerals. The idea of decimal place-value developed when the decimal scale was associated with the value of the places in a right to left order (or left to right) in the scale with the introduction of symbol for zero. That means, the symbols or word numerals took values × 1, × 10, × 100.........when placed in 1st, 2nd, 3rd places......respectively. From the beginning of the Christian era, the word-numerals with their various synonyms began to be used to avoid repetition of the same word and to keep rythm of ślokas.

A few synonyms of 0, 1, 2, 3, etc. may be given as follows :

0 *śunya*, *kha*, *ākāśa*, *ambara*, *vyoma*, *nabha*, *pūrṇa* etc.

1 *kṣiti*, *dharā*, *pṛthvī*, *bhū*, *bhūmi*, *indu*, *candra*, *abja*, *vidhu*, *śaśāṅka* etc.

2 *yama*, *aśvin*, *dasra*, *akṣi*, *netra*, *nayana*, *bāhu*, *kara* etc.

3 *rāma*, *guṇa*, *agni*, *bhuvana*, *loka* etc.

The application of nine word numerals on the decimal place-value scale was found in Sanskrit works and incriptions from 2nd century A.D. onwards. Several examples are given below :—

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1. Datta, B. 'Vedic Mathematics', *The Cultural Heritage of India*, old series, *3*, p. 395-97, Ramkrishna Centenary Memorial, Belur Math, Calcutta.

(1) *Pauliśasiddhānta* (200 A.D.)
kha-kha-rūpa-aṣṭa[1]
0 0 1 8 =8100.

(2) *Sūryasiddānta* (400 A.D.)
(i) *nava-vasu-sapta-aṣṭa-kha-nava-aśvi*[2]
9 8 7 8 0 9 2 =2908789.

(ii) *svara-eka-pakṣa-ambara-svara-ṛtu*[3]
7 1 2 0 7 6 =670217.

(3) *Pañcasiddhāntikā* (505 A.D.)
(i) *śunya-dvi-pañca-yama*[4]
0 2 5 2 =2520

(ii) *muni-yama-yama-dvi*
7 2 2 2 =2227

(4) *Inscription at Cambodia*[5] (604 A.D.)
rasa-dasra-śarai śakendra varṣe
6 2 5 =526 śaka year.

(5) *Inscription at Champa*[6] (731 A.D.)
rāma-artha-ṣaṭkai śaka
3 5 6 =653 śaka year

(6) *Inscription at Java*[7] (732 A.D.)
śruti-indriya-rasa śaka
4 5 6 =654 śaka

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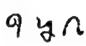
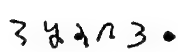
1. *Pañcasiddhāntika* of Varāha, ed. by Thibaut, ch. 6, Verse 9.
2. Vide *Pañcasiddhāntikā* of Varāhamihira (ed. by G. Thibaut) Ch. 9, v. 3.
3. *Ibid*, Ch. 9, v. 2.
4. Ch. 1, v. 17. (Thibaut's edition),
5. Coedès, G. 'A propos de l'origine des chiffres arabes', *Bulletin of the School of Oriental Studies*, *6*, p. 352, 1930-32.
6. Huber, E. 'Études Indo-Chinoises', *Bulletin de l'école Française, d'extrent orient*, *11*, p. 266-67, 1911.
7. Coedès, G. *Ibid.*, p. 326.

Such uses of word-numerals (i. e. same or synonymous words taking different values when placed at different places) are to be found in the *Brāhmasphuṭasiddhānta* and *Khaṇḍakhādyaka* of Brahmagupta (628 A D.) and in almost all later mathematical and astronomical works. The use of the word-numeral system is also found to occur in many of the inscriptions of Cambodia, Java, Champa, Malay etc. at about 7th century A. D.

(B) *Symbolic-use* : A good number of inscription using numeral symbols are also known. Some of these inscriptions used a point-symbol (*śunya-bindu*) as well as circular symbol for zero. The earliest evidence of the symbol for zero is found in the *Bakhshālī* Ms.[1] (400 A.D.). A chronological list of the appearance of the symbols of numerals from one to nine along with or without that of zero fitted in a decimal place-value scale, is given below :

(1) *Bakhshālī* Ms. (400 A.D.)

	Symbolic expression		*Number*
(i)		(17 verso)	330
(ii)		(22 recto)	157500
(iii)		(22 recto)	947
(iv)		(56 recto)	846720

1. Vide Hoernlé, A.F.R. 'The Bakhshālī Ms.', *Indian Antiquary, 17*, p. 33-48, 1888; Kaye, G. R. The *Bakhshālī* Ms., Parts I & II, New Imperial Series No. 43, Calcutta, 1927; Datta, B. 'The Bakhshālī Mathematics', *Bulletin of the Calcutta Mathematical Society, 21*, p. 1-60, 1929. The Bakhshālī Ms. was composed in the second century A. D. (Hoernlé, and Datta) and the present Ms. was probably a commentary.

(2) *Gurjara Grant inscription*[1] (595 A.D.)

 346

(3) *Khmere inscription at Sambor*[2] (683 A.D.)

 (śaka) 605 (śaka)

(4) *Malay inscription at Palembang*[3] (684 A.D.)

 (śaka) 606 (śaka)

 (śaka) 60 (śaka)

(5) *Malay inscription at Kotakapur*[4] (716 A.D.)

 (śaka) 608 (śaka)

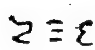
(6) *Dinaya Sanskrit inscription at Java*[5] (760 A.D.)

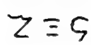
2 8 6

 (śaka) *nayana-vasu-rase* śake

=682 śaka

(7) *Inscription at Po-nagar Champa*[6] (813 A. D.)

(i)  (śaka) 735 (śaka)

(ii)  (śaka) 735 (śaka)

1. *Epigraphica Indica, 2*, p. 20, 1894.
2. Coedès, G., *Ibid., 6*, p. 326, Plate IV.
3. Westenenk, L. C. *Djāwā* periodical, *1*, no. 1, p. 10, vide plate I, vide also *Acta orientalia, 2*, p. 12-21, 1924.
4. Kern, H. Inscriptie van Kota Kapur, *Bidragen tot de Taal-. Land-en Volkenkunde van Nederlandische..., 67*, p. 393-400, 1913; Kern, H. *Verspreide Geschriften*, 7, p. 208-209, 1917; Coedès, G. *Ibid.*, plate IV.
5. Bosch, F. D. K. 'De Sanskrit inscriptie op den steen van Dinaja (682 śaka)', *Tijdschrift, 57*, p. 411, 1916; vide also *64*, p. 227-29, 1924.
6. Finot, L. 'Les inscriptions de Jaya Paramesvaravarman I, roi du Champa'. *Bulletin de l'école Française d'extrent orient, 15*, p. 47, 1915.

(8) *Inscription at Bakul*[1] (839 A. D.)

 (śaka) 751 (śaka)

We find here the use of word numerals with right to left order and symbolic numerals with a left to right order. The use of word numerals with left to right order has also been found in the *Mahāsiddhānta* of Āryabhaṭa II. These orders were so used for no obvious reason except possibly for emphasizing the idea that the decimal place-value concept remains unchanged in whatever order (right or left) the numerals are arranged when the number is always expressed by reading from higher to lower order in the scale. The symbolic use of zero is found to occur in the *Bakhshālī* Ms. (400 A. D.) of India as well as in number of inscriptions found at Sambor, Malay, Java, used in a decimal place-value system with their dates expressed in Śaka era of Indian origin. Some of these symbols including that of zero found in these inscriptions bear more or less close resemblance with those of the corresponding numbers in the *Bakhshālī* Ms. . The *Dinaya Sanskrit inscription* (see p. 72) found at Java (760 A. D.) however contained both word-numerals and the numeral symbols. This seems to suggest strongly a common origin of all these symbols obviously in India. Moreover, the evidence of the association of decimal scale e. g. *eka*, *daśa*, *śata* etc. with places is found in the works of Āryabhaṭa I (b. 476 A. D.) onwards. He writes that the value of the places are ten times of the preceding (*sthānāt-sthānaṃ daśaguṇaṃ syāt*, *Āryabhaṭīya*, Gaṇita, verse 2) in a numerical notation thereby expressing the idea of decimal place-value. That the decimal place-value concept was current before seventh century A.D.,

1. Finot, L. *Ibid.*, p. 47.

has also been attested by Vyāsa (before 700 A. D.) in his philosophical work, *Yogadarśanabhāṣya* (Ch 3, Sū. 13) thus : *yathaikā rekhā śatasthāne śataṃ daśasthāne daśa ekaṃ caikasthāne* i. e. "a (numerical) sign denotes hundred in the *śata* place, ten in the *daśa* place, and one in the *eka* place." The very similar language is used by Śaṅkarācārya (c. 800 A. D.) in his *bhāṣya* on *Vedānta Sūtra* (II. 2. 17). These illustrations establish without doubt that by the time of compositions of these two works, the decimal place-value concept must have been well established and was generally known. This system has appeared amongst the Arabs in the tenth century A. D. From there it went to Europe and it has now been adopted by all the civilised peoples of the world.

The Chinese (8th century A. D.) left a gap or some vacant space similar to the Babylonians where a zero was required. This is found on some Thang Mss. of the Tunhuang cave-temples. One of the Ms. rolls, namely *Li-Chhêng Suan Ching*[1] contains the number 405 both in written form and in rod numerals like |||| |||||. A symbol for zero in the usual circular form appeared only in 1247 A. D. in a work *Su Shu Chiu Chang* of Chhin Chiu-Shao.[2] Thus it gives : | ≡ ┬ 000 as 147000. Needham, from the consideration of amalgamation of Indian, Chinese and of the local residence of Java, Champa, Malay etc. due to colonisation, concluded that the origin of the symbol for zero should be attributed to the mixed culture of the Indians and the Chinese. But such an argument is quite untenable in view of the fact that the use of the symbol for

1. Needham, J. *Science and Civilisation in China*, *3*, p. 9, Cambridge, 1959.
2. Smith, D. E. *Hist. of Mathematics*, *2*, p. 42, Dover Edition.

zero fitted in a decimal place-value scale appears in Çhina only in Ms. of the 13th century A. D. Furthermore, there is no evidence that the use of word-numerals including that of zero in a decimal place-value scale was ever made by the Chinese or even by any other nations except by the Indian prior to their symbolic use. This has also been clearly recognised by the French scholar Coedès.[1]

There is no doubt that the Arabs derived their knowledge of decimal place-value from India. Al-Bīrūnī (1030 A. D.), who stayed in India quite a number of years and is very much well known for his encyclopaedical knowledge, made the following observations in his *India* (Eng. tr. Sachau, *1*, p. 174), "the numerical signs which we use are derived from the finest of the Hindu signs." The earliest use of zero in Arabia by a circular symbol was found in a 10th century A. D. manuscript.[2] In other Mss. during the same period, the forms such as φ, ᛗ and ⊃ were used for zero. The dot symbol (•), similar to that of the Indians, was also used by Bīrūnī in his table. The modern Arabic symbol 5 is represented by a symbol that looks somewhat like the modern zero and the zero is denoted by a dot. The Indian name *śunya* was taken over by the Arabs as *aṣ-ṣifr* or *ṣifr*. This was subsequently changed to zephirum (1202 A. D. - Fibonacci), *tziphra* (1340 A. D. - Planudes) and *zenero*, *zepiro* (c. 16th century A. D. Italy).[3]

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1. Coedès, G. *Ibid.*, *6*, p. 325.
2. Smith, D. E. *History of Math.*, 2, p. 74, Table Column III, Dover Edition.
3. Smith, D. E. *Ibid.*, p. 71.

The zero in its circular symbol appeared in Europe in a 13th century A. D. manuscript, reproduced in *Della vitā e delle opere di Leonardo Pisaro* (Rome, 1852). Next, it is found to have been used in European Manuscripts from 14th century A. D. onwards.[1]

Eight Fundamental Operations

The eight fundamental operations of Indian arithmetic after the invention of the decimal place value system of numeration are : addition (*saṃkalita*), subtraction (*vyavakalita* or *vyutkalita*), multiplication (*guṇana*), division (*bhāgahāra*), square (*varga*), square-root (*varga-mūla*), cube (*ghana*), cube-root (*ghana-mūla*).

These operations were carried out on a dust computing board (*pāṭī*). The method required the rubbing out of digits in every operation. According to Arabian scholars Āl-Khwārizmī (825 A. D.), Al-Uqlidisi (952 A. D.) Hindu arithmetic entered Islam with dust abacus as an intrinsic tool of it.[2] The latter author replaced the dust abacus to suit paper and ink. He used the Arabic word *takht* for *pāṭī*.

The ancient Indian mathematical works have not devoted much space to the methods of addition and subtraction in view of simple and self evident character, though such methods were described in details in the *Līlāvatī* of Bhāskara II (1150 A. D.) and in

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1. Smith, D. E. *Ibid.*, p. 71.
2. Datta, B. 'Testimony of early Arab writers on the origin of our numerals', *Bulletin of the Calcutta Mathematical Society*, *24*, p. 193-218, 1932; Saidan, A. S. 'The development of Hindu Arabic Arithmetic', *Islamic Culture*, *39*, p. 209-221, 1965.

two commentaries of *Līlāvatī* namely *Gaṇitāmṛta* of Sūryadāsa (1538 A. D.) and *Manorañjanā* of Rāmakṛṣṇa. Several methods of multiplication namely *kapāṭasandhi*, *gomūtrikā*, *khaṇḍa*, *bheda* and *iṣṭa* etc. were known. Datta and Singh have described this method fully in their *History of Hindu Mathematics*[1] as it appeared in the works of Mahāvīra (c. 850 A.D.), Śrīdhara (900 A.D.) and Śrīpati (1039 A.D.). The main feature of the method of *kapāṭasandhi* was that it was carried out in two stages: (1) the relative positions of the multiplicand and the multiplier followed by, (2) the rubbing out of figures of the multiplicand and substitution of the figures of the product in their places. The *kapāṭa* and *sandhi* owe their origin from the first and second stages respectively. The terms *hanana, vadha* or *kṣaya* which mean killing' or 'destroying' have been used in the works of Brahmagupta[2] (628 A. D.) in describing his process of multiplication. It seems to suggest that this method of *kapāṭasandhi* was known to India in 7th century A.D. This method reappeared in works of the Arabian scholars[3] Al-Nasabī (c. 1025 A. D.), Al-Ḥaṣṣār (c. 1175 A. D.), Al-Kalasādi (c. 1475 A. D.).

The method of division in India was given by Mahāvīra (850 A. D.), Śrīdhara (900 A. D.), Āryabhaṭa II (950 A. D.), Bhāskara II (1150 A. D.), Nārāyaṇa (1356 A. D.) and others.[4] The partial divisions were done and partial quotients were written in a separate line on the dust board and the

1. *1*, p. 134-149, 1935.
2. Brahmagupta mentions four methods of multiplication viz. *gomūtrikā, khaṇḍa, bheda* and *iṣṭa* (vide Datta and Singh, *Ibid.*, *1*, p. 135).
3. Datta and Singh, *Ibid., 1*, p. 143.
4. Datta and Singh, *Ibid., 1*, p. 150-154.

figures of these operations were obliterated. If the figures were not obliterated and the successive steps were written one below the other, this ancient process of division would become the modern method of long division. Āryabhaṭa I (499 A. D.) gave the method of extracting square root and cube root which depend on the method of division. This seems to suggest that the method originated in India about the 4th century A. D It reappeared in the Arabian works[1] from 9th century onwards. From Arabia, the method travelled to Europe where it came to be known as the galley (*galea, batello*) method. Fundamental operations for square and cube as developed by ancient and medieval Indian workers have been discussed fully by Datta and Singh.[2]

Square root and Cube root

Special credit is given to Āryabhaṭa I (c. 499 A.D.) for his rules for the extraction of the square and cube roots developed on the knowledge of decimal place value concept. His rule for square root extraction runs thus:

bhāgaṃ haredavargānnityaṃ dviguṇena vargamūlena |
vargādvarge śuddhe labdhaṃ sthānāntare mūlam ||
(*Āryabhaṭīya* ii. 4)

"Divide always the non-square (*avarga*) places by twice the square-root (up to the preceding odd place); after having subtracted the square (of the quotient) from the square place (*varga*), then the quotient placed at the separate place gives the root."

The other Indian scholars namely, Mahāvīra Śrīdhara, Āryabhaṭa II, Bhāskara II, Kamalākara

1. Smith, D. E. *History of Mathematics, 2*, p. 138-139, Dover Publication.
2. Datta and Singh, *Ibid., 1*, p. 155-169.

etc. have given the same method of extracting square root.[1] Āryabhaṭa I divides the number whose square-root is to be determined into *varga* (square) and *avarga* (non-square) places-respectively from right, while the other scholars have divided it into *sama* and *viṣama* places from right. The rule can be explained from the following illustration by putting □ and – marks against square and non-square places. Modern method of square root extraction is also given side by side for the purpose of comparison.

	Āryabhaṭa's method	*Modern Method*
Subtract the square 2^2 =	□ – □ – □ 5 4 7 5 6 (root 2 4 ... (1)	· · · 5 4 7 5 6 | 234 4 – – (1)
Divide by twice the root i. e., 2×2 = Subtract square of the quotient $3^2 = 9$	4)1 4 (3...(a) 1 2 2 7 9......(b) } root 23 (2)	43 | 1 4 7 1 2 9 ...... (2) 464 | 1 8 5 6 1 8 5 6..(3)
Divide by twice the root i. e., $2 \times 23 = 46$ Subtract square of the quotient i. e., $4^2 = 16$	1 8 5 (4......(a') 1 8 4 1 6 1 6......(b') } (3) ∴ the square-root is 234	

1. Singh, A. N. 'On the Indian Method of Root Extraction,' *Bulletin of the Calcutta Mathematical Society*, *18*, 123-40, 1927.

This shows that present method of extracting square root is same as that of Āryabhaṭa I (c. 499 A. D.) with the only difference that stages numbered (2) and (3) in the working of the modern method were carried out in two distinct steps.

The rule for the cube root extraction runs as follows :

aghanādbhajeddvitīyāt trighanasya mūlavargeṇa |
vargastripūrvaṃ guṇitaśśodhyaḥ prathamādghanaśca ghanāt ||

(*Āryabhaṭīya*, ii. 5)

"After having subtracted the cube (of the quotient) from the *ghana* place, divide the second *aghana* place by thrice the square of the root, and subtract the square of the quotient multiplied by thrice the previous root from the next (*aghana* place) and the cube (of the quotient) from the cubic place (next *aghana* place)."

The very same method is given by other Indian scholars like Brahmagupta, Mahāvīra, Śrīdhara, Āryabhaṭa II, Bhāskara II etc. Here also the number whose cube root is to be found is marked as—*ghana*, first *aghana*, and second *aghana*, from right.[1] For illustration, they may be denoted by the symbols say □ , —, — respectively. Āryabhaṭa's method of extraction is illustrated[2] in the following example; the modern method is also given side by side.

1. *prathama sthānaṃ ghanasaṃjñaṃ dvitīye tṛtīye aghanasaṃjñe | caturthaghanasaṃjñaṃ, pañcamaṣaṣṭhe aghanasaṃjñā ||*
(Parameśvara's Comm.)

2. The division is to be carried to such an extent that the steps (b) and (c) can be carried out (vide, p. 81).

	Āryabhaṭa I's method	*Modern method*
	1 9 5 3 1 2 5 (cube root 1	1953125 | 125
Subtract the cube $1^3 = 1$	1	1
	9	953
Divide by thrice the square of the root i.e. 3×1^2	3) 6 (2...... ...(a)	$1^2 \times 300 = 300$; $1 \times 30 \times 2 = 60$; $2^2 = 4$
	3 5	364 728
Subt. the square of the quotient multiplied by thrice the previous root i.e., $2^2 \times 3 \times 1 = 12$	1 2 ....(b)	225125
	2 3 3	$12^2 \times 300 = 43200$; $12 \times 30 \times 5 = 1800$; $5^2 = 25$
Subt. the cube of the quotient i.e., $2^3 = 8$	8....... (c)	45025 225125
	2 2 5 1	
Divide by thrice the square of the quotient i.e., $3 \times 12^2 = 432$	432)2 1 6 0(5...(a′)	
	9 1 2	
Subt. sq. of the quotient multiplied by thrice the previous root i.e., $5^2 \times 3 \times 12 = 900$	9 0 0....(b′)	
	1 2 5	
Subt. cube of the quotient i.e., $5^3 = 125$	1 2 5....(c′)	
	∴ cube root of 1953125 = 125.	

From the above it is clear that the modern cube root method of extraction is simply a contraction of Āryabhaṭa I's method.

Methods of extraction of square roots (*khai fang*) and cubic roots (*khai li fang*) with the help of abacus or counting boards appear in China in the *Chiu-Chang suan shu* (later half of the 1st century A. D.).[1] Yang Hui (c. 1261 A. D.) explained the methods of root extraction as found in the *Chiu-Chang suan shu* with an example of finding the square root of 71824 in his book, *Hsiang Chieh chiu chang suan fa* (A Detailed Analysis of the Mathematical Methods in the "Nine Chapters"). This gives the various stages on the counting board with the help of a diagram and illustrates the geometrical basis of the method.[2] In the cube-root method, Liu Hui and Li Shun-feng, commentators in the *Chiu Chang suan shu* both hinted at the use of cubical blocks to demonstrate the process. In Europe the derivation of the square-root method was done possibly from the Euclidean concept $(a+b)^2 = a^2 + 2ab + b^2$.

In the 4th century A. D. Theon Alexandria, explained Ptolemy's method of extracting square roots and found the square-root of 4500 degrees to be $67^\circ\ 4'\ 55''$ in sexagesimal unit. This method of extraction was purely geometrical and depended on Euclidean concept.[3] In another place he explained

1. Needham, J. *Science and Civilisation in China*, 3, p. 65–68, 1959.
2. Yong, Lam Lay. "The Geometrical Basis of the Ancient Chinese Square Root Method", *Isis*, **61**, No. 206, p. 92–101, 1970.
3. Heath, T. L. *A History of Greek Mathematics*, 1, p. 60–63, Oxford, 1921; Yong, Lam Lay, Ibid, p. 100–101.

$144 = 10^2 + 2.10.2 + 2^2 = (10 + 2)^2$ with the help of a diagram. This figure is two-dimensional and no attempt was made to construct a three dimensional figure in order to extend the square-root method to find the cube-root. In the sixteenth century A. D. both square-root and cube-root methods were given by Cataneo, which are exactly the same as those of Āryabhaṭa I[1]. Hence it is quite likely that the Indian methods of square-root and cube-root went to Europe through Arab intermediaries.

Numbers and Symbolism

Bhāskara I (600 A. D.) freely used integral and fractional numbers and their operations in his *Āryabhaṭīya-bhāṣya.* He has referred to in this work the names of mathematicians like Maskarī Pūrāṇa, Mudgala Patana and others. In his Āryabhaṭīya-bhāṣya[2], he has given ample illustrations of these operations as follows :

i) $x \pm \left(\frac{y}{z}\right) = \frac{(xz \pm y)}{z}$

ii) $\frac{x}{y} \pm \frac{z}{u} = \frac{xu \pm yz}{yu}$

iii) $\frac{x}{y} \times \frac{z}{u} = \frac{xz}{yu}$

iv) $\frac{x}{y} \div \frac{z}{u} = \frac{xu}{yz}$

1. Smith, D. E. *History of Mathematics*, 2, Dover Publication, p. 148.
2. Shukla, K. S. 'Hindu Mathematics in the seventh century as found in Bhāskara I's, commentary on the *Āryabhaṭīya*', *Gaṇita*, 22, No. 2, p. 61-63, 1971.

v) $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$

vi) $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$

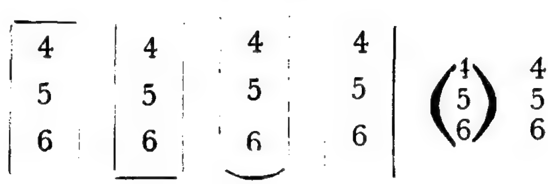
vii) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

viii) $\sqrt[3]{\frac{x}{y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{y}}$

It is not clearly known to what extent he was indebted to these scholars since their works are lost.

Bhāskara I wrote integral numbers within square or circular cells and sometimes without use of any cells The numbers 4, 6 and 7 without circular are written in any one of the following forms.

The fractional numbers are sometimes enclosed within rectangular cells or brackets without sticking to any uniform pattern. Thus the number $4\frac{5}{6}$ was written in any one of the following way.



The enclosures were perhaps meant to avoid confusion and mixing up of numbers with the written matter.

Symbol for Minus Sign

In the *Bakhśālī* Ms , the modern plus sign (+) has been used to represent minus sign. Bhāskara I has made use of a little circle (•) on the right of the numbers to be subtracted. For example,

$$\left(\frac{1}{2}\right) - \left(\frac{1}{6}\right) \text{ is written as } \frac{1}{2}\ \frac{1}{6}\cdot$$

The plus sign, to denote negative sign, is also used in Jaina work *Dhavalā*, a commentary on the *Ṣaṭkaṇḍāgama* (vol. 10, p. 151). The first letter *ri* of the word *rina* was written in Brāhmī script as ϒ . Perhaps the sign ϒ was changed later into + to denote negative sign. But how little circle for negative sign came to be used, is not known. Another manuscript,[1] a commentary on the *Pāṭīgaṇita* of *Śrīdhara*, available at Kashmir, has used the + symbol for the minus sign. The symbol + is generally used after the number affected, but sometimes it was also used before the number affected. Thus, – 3 is written as 3 + , and sometimes as + 3.

This Kashmirian manuscript has used dot (.) as well as (o) for zero. The dot symbol was also found to have been used in the *Bakhśālī Ms.*[2] (c. 400 A. D.). The dot as a symbol for zero is found to be more ancient than the small circle, and this shows that the mathematicians of north-west region of India had not yet given up the use of dot, the alternative symbol for zero.

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1. Ed. by K. S. Shukla, Lucknow, 1959.
2. The absence of the use of a small circle as a symbol for zero testifies that it is a much older work than the present commentary.

Rule of Three and its Inverse, Five, Seven, etc,

The term used in ancient India for rule of three is *trairāśika,* for three things were involved in the problems. It has appeared in the *Bakhshālī Ms.* (400 A. D.), *Āryabhaṭīya* of Āryabhaṭa I and in all other later mathematical works.[1] The method gives the solution of the problem like : if p yields f, what will i yield ? where p = *pramāṇa* (argument), f = *phala* (fruit) and i = *icchā* (requisition). According to Śrīdhara, the *pramāṇa* and *icchā,* which are of the same demominations should be set down in the first and last places; the *phala* which is of different denominations, be set down in the middle as follows :

p	f	i

For required result the middle quantity is to be multiplied by the last quantity and divided by the first quantity.

i.e. required result = $\frac{f \times i}{p}$

Example "A certain lame person goes to a distance of $\frac{1}{8}$ of a yojana in $\frac{1}{3}$ of a day. Say in how much time will he go to a distance of 100 yojanas ?".[2]

Here p = $\frac{1}{8}$ of a yojana.
f = $\frac{1}{3}$ of a day.
i = 100 yojanas.

1. *Āryabhaṭīya,* ii, 26; *Brahmasphuṭa-Siddhānta,* xii. 10, *Gaṇitasāraśaṃgraha,* v. 2 (i); *Mahāsiddhānta,* xv. 24–25; *Gaṇitatilaka,* v. 86, p-68; *Siddhānta Śekhara,* xiii, 14; *Gaṇitakumudī* I, v. 60, p. 47; *Pāṭīsāra* of Muniśvara, v. 83.
2. *Gaṇitasārasaṃgraha,* v. 3, 4; *Gaṇitatilaka* v. 93, p. 72; *Pāṭīgaṇita,* Ex. 30, p. 24 (Eng. tr.)

As per direction in the rule, we write

$\frac{1}{8}$	$\frac{1}{3}$	100

$$\text{required result} = \frac{\frac{1}{3} \times 100}{\frac{1}{8}} = 266\tfrac{2}{3} \text{ days}$$

$$= 8 \text{ months and } 26\tfrac{2}{3} \text{ days}$$

In the Renaissance period, this rule was highly appreciated to solve commercial problems by Widman (1489 A. D.). Tonstall (1522 A. D.), Gamma Frisius (1540 A. D.), Recorde (c. 1542 A. D.) etc. Recorde calls this rule, 'the rule of proportion', sometimes 'Golden Rule' for its excellency in arithmetical computation.[1]

For inverse rule of three[2] (*vyasta-trairāśika*), there is change in the units of measurment. The required result in this case is obtained when the middle quantity is multiplied by the first quantity and divided by the last quantity i.e. $\frac{f \times p}{i}$. Āl-Bīrūnī wrote a treatise, *Fi raśikat al-hind* ('rule of three or more terms of the Hindus'). In his *India* (*1*, p. 313), he has used an example of *vyasta-trairāśika* (Inverse Rule of Three). The rule was known to the Arabs from the eighth century onward. It is quite likely that the rule went to Europe through the Arabs where it was known later as 'the Golden Rule'.

The rule of five, the rule of seven, the rule of nine etc. were so named, for there are five, seven, and nine

1. Smith, D. E. *History of Mathematics*, Dover Publication, 2, p. 484.
2. *Brāhmasphuṭasiddhānta*, xii, 11,

terms etc. involved in the problems. By these rules, the problems of principal and interest, prices of various types of gold, calculation of charges for carring weights to a distance, barter of commodities, sale of living things etc. are solved. The two sides referred to in these rules are known as *pramāṇarāśi-pakṣa* (the argument side), and *icchārāśi-pakṣa* (the requisition side). After placing the given numbers in these two sides, the fruit has to be transposed from one side to the other and then denominator of two sides are interchanged in like manner and each side is multiplied. The required result is obtained when the side with larger quantities is divided by the side with smaller quantities. This can be clarified from the following example.[1]

Example. "If $1\frac{1}{2}$ be the interest on $100\frac{1}{2}$ for one-third of a month, what will be the interest on $60\frac{1}{4}$ for $7\frac{1}{2}$ months?"

Here, the two sides may be written as,

pramāṇarāśi pakṣa (a)	*icchārāśi pakṣa* (b)
$100\frac{1}{2}$ or $\frac{201}{2}$	$60\frac{1}{4}$ or $\frac{241}{4}$
$\frac{1}{3}$	$\frac{15}{2}$
$1\frac{1}{2}$ or $\frac{3}{2}$	o

1. *Gaṇitasārasaṃgraha* v. 34; *Gaṇitatilaka*, v. 99, p. 76; *Gaṇita-Kaumudī*, I, lines 6–9, p. 51; Similar examples appeared in Bhāskara I's commentary on the *Āryabhaṭīya*, ii, 26–27.

The symbol o is written for the desired quantity. In the *Pāṭīsāra* (v. 88–92) of Muniśvara[1] the place for desired quantity is left vacant. Next, the fruit is transposed and denominators are interchanged as follows :

(a)	(b)
$\frac{201}{2}$	$\frac{241}{4}$
$\frac{1}{3}$	$\frac{15}{2}$
o	$\frac{3}{2}$

(a)	(b)
$\frac{201}{4}$	$\frac{241}{2}$
$\frac{1}{2}$	$\frac{15}{3}$
$\frac{o}{2}$	3

The required result

$$= \frac{241 \times 2 \times 15 \times 3 \times 3}{201 \times 4 \times 1 \times 2 \times 2} = 20\frac{125}{536}$$

Commercial Problems

The ancient Indian scholars dealt with various commercial problems. Of these, the problems on mixture (*miśraka vyāvahāra*) are interesting, for it relates to many problems on capital and interest (when the amount and rate of interest given), rules for finding time (for a sum lent out at simple interest and paid back in equal monthly instalments), *varṇa* of an alloy, weights, shares of partners, wages and payments etc. besides many others. The solution of these problems sometimes require the knowledge of quadratic equations. There are also some problems which are of indeterminate types. A few may be discussed in what follows:

1. Manuscript in the Govt. Sanskrit College Library, Benaras.

Problem 1. "The rate of interest being 5 percent per month, a certain sum is amounted to 96 in a year. Say friend, what is the capital (*mūla*) and the interest (*vṛddhidhana*)".[1]

Here argument (*pramāṇarāśi*) = p = 100 (say), time (*nija-kāla*) = t = 1 month, fruit (*phala*) = f = 5, amount = capital (*mūladhana*) + interest (*vṛddhi-dhana*) = a = 96, other time (*parakāla*) = n = 1 year = 12 months.

The rules for finding the capital and interest are :

$$\text{Capital} = \frac{p.\ t.\ a}{p.\ t + f.n}, \quad \text{Interest} = \frac{f.\ n.\ a}{p.t + f.n}$$

$$\therefore \text{Capital} = 60, \quad \text{Interest} = 36.$$

Problem 2. "The rate of interest being 5 percent per month, the commission of the surety (*bhāvyaka*) 1 percent per month, the fee of the calculator (*vṛtti*) $\frac{1}{2}$ per cent per month, the charges of the scribe $\frac{1}{4}$ percent per month, a certain sum amounts to 905 in a year. (Find the capital, interest, and the shares of the surety, calculator and the scribe)."[2]

Here $p = 100$, $t = 1$, $f_1 = 5$, $f_2 = 1$,

$f_3 = \frac{1}{2}$, $f_4 = \frac{1}{4}$, $a = 905$, $n = 1$ year $= 12$

$$\text{Capital} = \frac{p.\ t.\ a}{p.t + (f_1 + f_2 + f_3 + f_4) \times n.} = 500$$

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1. *Gaṇitatilaka*, v. 112, p. 83; *Pāṭīgaṇita*, Ex. 52; similar examples occur in *Gaṇita Kaumudī*, 1, lines 10–11, p. 60, Pṛthudaka's comm. on *Brāhmasphuṭasiddhānta* x ii. 14.
2. *Gaṇitatilaka*, v. 114, p. 83; *Pāṭīgaṇita* Ex-54.

$$\text{Interest} = \frac{f_1 \; n. \; a}{p.t + (f_1 + f_2 + f_3 + f_4). \; n} = 300$$

Commission of surety

$$= \frac{f_2. \; n. \; a}{p.t + (f_1 + f_2 + f_3 + f_4). \; n} = 60$$

Similarly the fee of the calculator = 30

and the charges of the scribe = 15

Problem 3. "What in the *varṇa* when (three pieces of gold of) $\frac{16}{3}$, $\frac{25}{6}$, $\frac{15}{2}$ *māṣas*, having $\frac{23}{2}$, 10, $\frac{15}{2}$ *varṇas* (respectively) are mixed in one ?"[1]

Here $w_1 = \frac{16}{3}$, $w_2 = \frac{25}{6}$, $w_3 = \frac{15}{2}$ and $v_1 = \frac{23}{2}$, $v_2 = 10$, $v_3 = \frac{15}{2}$. The mixed *varṇa* v of the alloy is given by the rule as follows.[2]

$$v = \frac{v_1 \; w_1 + v_2 \; w_2 + v_3 \; w_3 + \ldots + v_n \; w_n}{w_1 + w_2 + \ldots + w_n}$$

$$\text{here} \quad v = \frac{\frac{23}{2}.\frac{16}{3} + 10.\frac{25}{6} + \frac{15}{2}.\frac{15}{2}}{\frac{16}{3} + \frac{25}{6} + \frac{15}{2}}$$

$$= \frac{1911}{204} = 9\frac{75}{204} = 9\frac{25}{68}.$$

Problem 4 : "There are two small balls of gold of equal worth (i e. having equal quantities of pure gold, whose combined weight is 5 *māṣas*. When they are respectively combined with $\frac{2}{3}$ part and $\frac{1}{2}$ part of the other, they become 10 and 9 *varṇas* respectively. (Find their weights and *varṇas* respectively.)"[3]

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1. *Pāṭīgaṇita*, Ex. 62; Similar examples occur in *Bakhshali* Ms. 16 verso, 17 recto and 17 verso; *Gaṇitasārasaṃgraha*, vi, 170–171.
2. C/o. *Bakhśāli Ms.* 16 verso; *Gaṇitasārasaṃgraha*, vi. 169; *Pāṭīgaṇita*, Rule 52; *Gaṇita Kaumudī*, I, lines 5–7, p. 76.
3. *Pāṭīgaṇita*, Ex. 69.

Let w_1 and w_2 be the weights of the two gold balls having *varṇas* v_1 and v_2.

here $w_1 + w_2 = 5$......(1) $w_1 v_1 = w_2 v_2$......(2)

$$v_1 = \frac{(w_1 + \frac{2}{3} w_2)}{(1 + \frac{2}{3}) w_1} \cdot 10 \ldots \quad (3)$$

$$\text{and } v_2 = \frac{w_2 + \frac{1}{2} w_1}{(1 + \frac{1}{2}) w_2} \cdot 9 \quad \ldots \quad (4)$$

The results (1), (2), (3) and (4), give $w_1 = 2$ *māṣas*, $w_2 = 3$ *māṣas* and $v_1 = 12$ and $v_2 = 8$.

The results (3) and (4) were known also to other scholars.[1]

Problem 5 : "Pigeons are sold at the rate of 5 for 3 (*rūpas*), cranes at the rate of 7 for 5 (*rūpas*), swans at the rate of 9 for 7 (*rūpas*), and peacocks at the rate of 3 for 9 (*rūpas*). Knowing the rates as stated above, bring 100 birds for 100 *rūpas* for the amusement of the prince".[2]

This is an indeterminate problem involving large number of solutions and occurs in the *Gaṇitasāra-saṃgraha* (vi, 152–53), *Pāṭīgaṇita*, (Ex. 78–79); *Bījagaṇita* of Bhāskara II (Ānandāśrama Sanskrit series p. 163), and *Gaṇitakaumudī* of Nārāyaṇa (p. 1,

1. C/o. *Gaṇitasārasaṃgraha*, vi, 209–12, *Pāṭīgaṇita*, Rule 58, *Gaṇita Kaumudī*, 1, lines 9–18, p. 81.

2. A similar problem occurs in the *Bakhśālī Ms.* III, 58 verso which runs as follows : "A man earns 3 *maṇḍas* in a day, a woman 1½ *maṇḍas* in a day, and a *sūḍha* ½ *maṇḍa* in a day. If 20 of them earn 20 *maṇḍas* in a day, how many of each category are there ?".

lines 2–5, p. 93) The problem leads to the solution of :

$$5x + 7y + 9z + 3t = 100$$

$$3x + 5y + 7z + 9t = 100$$

where (5x, 3x), (7y, 5y), (9z, 7z) and (3t, 9t) are the number and price of the pigeons, cranes, swans, and peacocks. Shukla has given 16 integral solutions of which four such solutions were given earlier by the commentator of Śrīdharācārya.[1]

Factors of Positive Integer

The ordinary method of factorisation by successive division by 2, 3, 5 etc, was well-known in ancient India. The rule for the first time, occurs in the *Siddhāntaśekhara* of Śrīpati (1039 A. D.) and also appear in later works.[2] In addition to this method he gives also rule for factorising a non-square number, as follows :

Rule I. "When the dividend is even, it is divided by 2 again and again, and divided by 5 when 5 occurs in the unit place; this is continued until the dividend is reduced to an odd number; then the (prime number) 3 etc are to be tried as divisors".[3]

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1. *Pāṭīgaṇita*, Ed. by Shukla, p. 51.
2. Shukla, K. S. "Hindu Methods for finding factors or divisors of a number", *Gaṇita*, 17, No. 2, pp. 109–117. 1966.
3. *dvābhyāṃ dvābhyāṃ bhājyarāśiṃ ,*
   same tadādisthāne pañcake pañcakena ı
   evaṃ kuryādyāvadojaṃ tu tāvat ,
   tryādyaihārairbhājyarāśiṃ bhajettu ıı
   (*Siddhāntaśekhara*, xiv, 36–37 (Babuaji Miśra's Edition)

Rule II. "When the dividend is a perfect square, its square root itself is a divisor, if it is not a perfect square its nearest square root is multiplied by 2, increased by 1 and diminished by the residue of the square root, then it is a perfect square. The square root of this perfect square and also the square root of the dividend as increased by this perfect square, added and subtracted, (will give the two factors of the dividend)".[1]

Besides Rule I, Nārāyaṇa (1356 A. D.) gives a fuller treatment of the Rule II in his *Gaṇitakaumudī* rediscovered later by French mathematician Fermat[2] in 1643 A. D. Nārāyaṇa's rule runs as follows. "The nearest square root of the non-square number, is multiplied by 2, increased by 1, and diminished by the residue of the square root. If it is a perfect square, it is added to the dividend to make it a perfect square. If the above number is not a perfect square, it should be increased by the successive terms of the arithmetic series whose first term is twice the nearest square root (of the non-square number) increased by 3 and common difference 2, until it becomes a perfect square. The unequal (successive) divisors thus obtained are multiplied by the preceding ones and by the product of the preceding ones taking one, two, three......all at a time and are placed ahead of those divisors. In case of equal divisors, each is multiplied by the (pro-

1. *vargaścettanmūlamevāsya hāraṃ,*
no ced āsannaṃ padaṃ dvighnamasmin ।
rūpaṃ yuktvā śeṣahīne kṛtiḥ syāt-
tanmūlaṃ tadyuktamūlaṃ yutone ॥
(*Siddhāntaśekhara*, xiv, 36 37).

2. Dickson, L. E. *History of the Theory of Numbers*, 1, p. 357.

duct of the preceding ones. Each of these is to be multiplied by the others (to get the divisors of the given number".[1]

The rule 2 in symbols may be expressed as follows :

Let $N = a^2 + r$

if $2a + 1 - r = b^2$

than $N + b^2 = a^2 + r + (2a + 1 - r) = (a + 1)^2$

so that $N = (a + 1)^2 - b^2$

$= (a + b + 1)(a - b + 1)$

If however, $2a + 1 - r = c$, where c is not a perfect square, then some terms of the series $(2a + 3)$, $(2a + 5)$, $(2a + 7)$......are added to c, so as to make the resulting sum a perfect square.

Let $c + [(2a + 3) + (2a + 5) + \ldots r \text{ terms}] = k^2$

or $c + (2a + 3) + (2a + 5) + \ldots + (2a + 2r + 1) = k^2$

Then $N + k^2 = (a + r + 1)^2$

or $N = (a + r + 1)^2 - k^2$

$= (a + r + k + 1)(a + r - k + 1)$

Last part of the rule says if $N = p \times p \times p \times q \times s \times t$ then the divisiors of N are :

q, s, t, qs, qt, st, qst; p, p^2, p^3; qp, qp^2, qp^3 sp, sp^2, sp^3, tp, tp^2, tp^3, qsp, qsp^2, qsp^3, qtp, qtp^2, qtp^3, stp, stp^2, stp^3, qstp, $qstp^2$, qstp.[3]

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1. *apadapradasya rāśeḥ padamāsannaṃ dvisaṃ guṇaṃ saikaṃ* ॥ *mūlāvaśeṣahīnaṃ vargaścet kṣepakaśca kṛtisiddhayai* । *vargo na bhavet pūrvāsannapadaṃ dviguṇitaṃ trisaṃyuktamādyād dyuttara vṛddhyā tāvad yāvad bhaved vargaḥ* । *asamānānāṃ pūrvahatāḥ pare puraḥsthāstathā cānye tulyānāṃ pūrvaghnaḥ paraḥ pṛthak te'nyaharanighnāḥ* ।

(*Gaṇitakaumudī*, II, xi, 6½ – 8½, pp. 245-47, Padmākara Dvivedī's edition).

The following two example used by Nārāyaṇa will illustrate the above the rule.

Example 1. "O expert in mathematics, tell me if you know the subject of finding divisors, the numbers by which the number 1161 is exactly divisible"

$$\text{Now } 1161 = 34^2 + 5$$
$$\text{and } 2 \times 34 + 1 - 5 = 64 = 8^2$$
$$\therefore 1161 + 8^2 = 35^2$$
$$\text{or } 1161 = 35^2 - 8^2$$
$$= 43 \times 27 = 3 \times 3 \times 3 \times 43$$

∴ Possible divisors of 1161 are :

3, 9, 27, 43, 129, 387, 1161

Example 2 : "If you are fully proficient in finding divisors, quickly tell me the numbers by which 1001 is exactly divisible"

$$\text{Now } 1001 = 31^2 + 40$$

and $2 \times 31 + 1 - 40 = 23$, not a perfect square.

$\therefore \; 23 + (2.31 + 3) + (2.31 + 5) + \ldots\ldots = k^2$, say.

i.e. $23 + 65 + 67 + \ldots\ldots + 89 = k^2 = 32^2$.

hence by the rule,

$$1001 + k^2 = 1001 + 32^2 = 45^2$$

or, $1001 = 45^2 - 32^2 = 77 \times 13 = 7 \times 11 \times 13$

The possible divisors are :

7, 11, 13, 77, 91, 143, 1001

Surd Number

In the *Śulbasūtra*[1], a surd was technically called *karaṇī*. Thus *dvikaraṇī* means $\sqrt{2}$, *trikaraṇī* $= \sqrt{3}$, *catuṣkaraṇī* $= \sqrt{4}$, *saptamakaraṇī* $= \sqrt{\frac{1}{7}}$ etc. By *karaṇī*,

1. Datta, B. *The Science of śulba*, University of Calcutta, p. 188, 1932.

root of both square and non-square numbers were meant. The idea of terms 'audible' for rational number and 'inaudible' for irrational number used by the Arabian scholar al-Khwārizmī (c. 825 A.D.) perhaps developed from Hindu term *karaṇī*[1]. The *Śulba* authors knew the operations of addition and multiplication on surd numbers. Only in one instance, *Śulba* authors viz., Baudhāyana, Āpastamba and Kātyāyana calculated the value of $\sqrt{2}$ in terms of fraction by following a method of approximation. It has been discussed in the chapter on geometry. This is :

$\sqrt{a^2+r} = a + \frac{r}{2a+1} + e_1$ where e_1 is the third term approximation.

The formula $\sqrt{A} = \sqrt{a^2+r} = a + \frac{r}{2a}$ where a = greatest root, r = original minus the square of the greatest root = $A - a^2$, was given by a Jaina canon writer (1st century A.D.)[2] The *Bakhśālī* Manuscript[3] (4th century A.D.) gave a more accurate formula for finding an approximate value of the square root of a non-square number as follows :

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1. The term *karaṇī* was however changed to *karṇa* meaning 'ear' which may be classified as 'bad ear' and 'good ear'. 'Bad ear' means 'inaudible' i.e., where root cannot be exactly determined and 'good ear' means 'audible' whose root can be determined.

2. Datta, B. 'Jaina School of Mathematics' *Bulletin of the Calcutta Mathematical Society*, 21, p. 132, 1929.

3. Datta, B. 'The Bakhshālī Mathematics' *Bulletin of the Calcutta Mathematical Society*, 21, p. 11-12, 1929.

$$\sqrt{A} = \sqrt{a^2+r} = a + \frac{r}{2a} - \frac{(\frac{r}{2a})^2}{2(a+\frac{r}{2a})} \quad \text{(approx.)}$$

By following this, $\sqrt{41}$ is found as $\sqrt{41} = 6 + \frac{5}{12} - \frac{(\frac{5}{12})^2}{2(6+\frac{5}{12})}$. This formula is also used to calculate the values of $\sqrt{41}$, $\sqrt{105}$, $\sqrt{487}$, $\sqrt{889}$, $\sqrt{339009}$. The rule is attributed to Greek Heron (200 A.D.) and is restated by al–Haṣṣarār (c. 1175 A.D. ?) and medieval Arabian algebraists.[1]

Bhāskara I has made use of surds in his *Āryabhaṭīya-bhāṣya.*

He knew, $\sqrt{x}.\sqrt{y} = \sqrt{xy}$

$x.\sqrt{y} = \sqrt{x^2y}$

$\sqrt{xy}+\sqrt{yz} = \sqrt{y}(\sqrt{x}+\sqrt{z})$ etc. He has admitted that the rules on surds has been quoted from earlier works. For approximate value of non-square number, Śrīdhara[2] and Munīśvara (*Pāṭīsāra* v. 117) used the following formula,

$$\sqrt{A} = \sqrt{\frac{A.\ a^2}{a^2}} \quad \text{(where a is a large number chosen)}$$

1. Heath, T. L. *History of Greek Mathematics*, 2, p. 324; Smith, D. E. *History of Mathematics*, Dover Publication, 2, p. 254 (f. n.).
2. *rāśermūladasyā hatasya vargena kenacit mahatā* ।
mūlaṃ Śeṣeṇa vinā vibhajet guṇavarga–mūlena ।
(Sūtra 118)

Similar rules are also available in the *Mahāsiddhānta* (xv. 55), *Siddhāntaśekhara* (xiii. 36), *Gaṇita Kaumudī*, BK. II, p. 33, lines 4-7; and *Paṭīsāra*, v. 117.

$$\text{or} \quad \sqrt{A} = \frac{\sqrt{p^2+r}}{a} - \frac{p}{a}$$

For example, $\sqrt{10} = \sqrt{\frac{10 \times 49}{49}} = \sqrt{\frac{22^2+6}{7}} = \frac{22}{7}$(approx.)

Bhāskara II (1150 A.D.) used the following formula:[1]

$$\sqrt{A} = \sqrt{a+r} = \frac{a + \frac{A}{a}}{2}$$

The same rule was used by some Arabian scholars[2] in the 14th century A. D. to approximate the value of $\sqrt{10}$. Bhāskara II[3] (1150 A. D.) gave also methods of extracting the square root of surd quantities like $a+\sqrt{b}$, $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ where a, b, c, d are rational. He considered also the problem of the type $\sqrt{10+\sqrt{24}+\sqrt{40}+\sqrt{60}} = \sqrt{2}+\sqrt{5}+\sqrt{3}$; and many others. Bhāskara II, Nārāyaṇa and others perhaps, also knew to calculate the approximate values of the surd quantities like $\sqrt{N}$ by the method of continued fraction (vide chapter on Algebra).

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1. Chakravarti, G. 'Surd in Hindu Mathematics', *Journal of the Department of Letters*, Calcutta University, 24, p. 29–58, 1934.
2. Smith, D. E. *History of Mathematics*, Dover Publication, 2, p. 259. The relation $\sqrt{A} = a + \frac{A-a^2}{2a}$ was used by the Arab scholar. This is equivalent to $\sqrt{A} = \frac{a+\frac{A}{a}}{2}$
3. Vide his *Bījagaṇita*—Section on *Karaṇī*, See also Srinivasiengar, C. N. *The History of Ancient Indian Mathematics*, p. 90, World Press, 1967.

Nārāyaṇa[1] (1357 A. D.), however, has given the following method :

mūlaṃ grāhyaṃ yasya ca rupakṣepaje pade tatra |
jyeṣṭhaṃ hrasvapadenoddharenamūlamāsannam |

(*Bījagaṇita* of Nārāyaṇa, Bk. 1, Rule 88)

"Obtain the roots (of a square nature) having unity as the additive and the number whose square root is to be determined (as the multiplier). Then the greater root divided by the lesser root will be the approximate value of the square root."

If $x = a$, $y = b$ be a solution of $Ax^2 + 1 = y^2$, then

$$\sqrt{A} = \frac{y}{x} = \frac{b}{a} \text{ approximately.}$$

Nārāyaṇa calculated the values of $\sqrt{10}$ and $\sqrt{\frac{1}{5}}$ from $10x^2 + 1 = y^2$. and $\frac{1}{5}x^2 + 1 = y^2$ to give illustrations of this method.

In the first example, according to Nārāyaṇa, "the roots for the additive unity are (6, 19), (228, 721),

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1. Since $Aa^2 + 1 = b^2$, we have $\sqrt{A} = \sqrt{\frac{b^2-1}{a^2}} = \frac{b}{a}\sqrt{\left(1-\frac{1}{b^2}\right)}$.

If b is large, $\frac{b}{a}$ is a close approximation to $\sqrt{A}$. The error is less than $\frac{1}{2x^2\sqrt{N}}$ and greater than $\frac{1}{2xy}$. This can be verified at once by writing $\sqrt{A} = \frac{y}{x}\sqrt{1-\frac{1}{y^2}}$ and expanding the right hand side by the binomial theorem.

(8658, 27379). Then dividing the greater root by the smaller root the approximate values are :

$$\sqrt{10} = \frac{19}{6}, \frac{721}{228}, \frac{27379}{8658}$$

likewise, $$\sqrt{\frac{1}{5}} = \frac{9}{20}, \frac{161}{360}, \frac{2889}{6460}.$$

These are arbitrary solutions, for how $x = a$ and $y = b$ were selected for solution of $Ax^2 + 1 = y^2$ is not clearly pointed out. However, a similar method of closer approximations to the value of surd was given by Euler[1] in 1732 A. D. Nārāyaṇa's method was not followed by Jñānarāja[2] (1503 A. D.) who for approximating the value of $\sqrt{N}$, found the nearest perfect square a^2 to A and formed closer approximation as follows :

$a_1 = \frac{1}{2}\left(a + \frac{A}{a}\right)$. A next approximation $a_2 = \frac{1}{2}\left(a_1 + \frac{N}{a_1}\right)$ is formed and so on. Jñānarāja illustrated his rule by evaluating $\sqrt{8}$, $\sqrt{20}$, etc.

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1. Datta, B. "Nārāyaṇas's Method for values of Surds". *Bulletin of the Calcutta Mathematical Society*, 23, p. 187–194, 1931.

2. *āsannamūlena hatātsvavargāllabdhena mūlaṃ sahitaṃ vibhaktaṃ bhavettadāsannapadaṃ tato'pi muhurmuhuḥ syātsphuṭa mūlameva* (*Siddhāntasundara* of Jñānarāja, the chapter on Algebra).

"Divide its square by the root of the nearest square number, the quotient together with that approximate root being halved will be a root more approximate than that. Values more and more accurate can certainly be found by proceeding in the same way repeatedly"

The rule of Jñānarāja has been reproduced by his son Sūryadāsa in his commentary on the algebra of Bhāskara II. It reappears in the works of Kamalākara[1] (1658 A. D.). From the standpoint of actual computation, the approximate values of surds as obtained by Jñānarāja were stated by Greek Heron[2] (200 A. D.), who however gave no method of his calculation.

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1. *svāsannamūlena hatādvargā labdhisṭadāsannapadaikya khaṇḍam* ।
yat tat sphuṭāsannapadena sūkṣaṃ tadvat sphuṭasannapadaṃ muhuḥ syāt ॥
(*Siddhāntatattvaviveka*, Spaṣṭādhikāra, v. 14).

2. Heath, T. L. *History of Greek Mathematics*, 2, p. 324. Heath mentions, some of the mathematicians of the middle ages and of modern times have applied the same processes.

CHAPTER III

GEOMETRY

Unlike Greek geometry which attained independent existence with definitions, axioms, theorems and proofs arranged in logical order, the geometry developed in India was of intuitive nature, which, like those of the old Egyptians and Romans consisted cnly of empirical rules. It gathered facts relating to geometrical problems without attempting to solve them by deductive reasoning. The demonstrative geometry was the central feature of Greek mathematics culminating in Euclid (c. 300 B. C.) and Apollonius (c. 200 B. C.), whereas Indian geometry like that of the Chinese lay rather in the direction of algebra.

The sacrifice was the prime religious avocation of the Vedic Hindus. The culture of astronomy was chiefly needed to fix the proper time for the sacrifice. The *Kalpasūtra*[1] contains, besides other matters of ritual, the rules for the proper construction of sacrificial altars, It was perhaps in this connection that the study of problem of geometry as also arithmetic and algebra began in ancient India. Geometry was then the science of altar construction. The *Śulbasūtra* (meaning

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1. The *Kalpasūtras* form one of the six *Vedāṅgas*. They are broadly divided into two classes viz. the *Gṛhyasūtras*, the rules for ceremonies relating to family or domestic affairs, and the *Śrautasūtras*, the rules for ceremonies ordained by the Veda such as the preservation of sacred fires, performance of sacrifices etc.

'the rules of the cord') is a part of *Kalpasūtra*. The *Śulbasūtras* of Baudhāyna, Āpastamba, Katyāyana, and Mānava are the main sources of our knowledge of geometry[1]. We thus find that in Vedic India, the astronomy in one hand and geometry on the other were cultivated under different circumstances by different classes of priests having different duties apportioned to them.

Vedic Altars

Datta in his book, *the Science of Śulba*,[2] has traced the growth and development of the geometry of this period. Various types of altars are described by him as available in the different *Saṃhitās*, *Brāhmaṇas* and especially in the *Śulbasūtras*. These altars are mostly brick structures of considerable size and shape. They are mainly *Āhavanīya*, *Gārhapatya*, *Dakṣināgni*, *Mahāvedi* (also called *Saumiki vedi*), *Sautramāṇi vedi*, *Prāgvaṃśa*, *Śyenacit*' *Vakrapakṣa vyastapuccha śyena*, *Kaṅka*, *Alaja*, *Prauga*, *Ubhayataprauga*, *Rathacakracit*, *Droṇa*, *Samuhya*, *Paricāyya*, *Śmaśāna*, *Kūrmacit* etc. Besides many other duties, the *Śulbakāras* had to perform several other important duties systematically viz.

1. The names of nine *Śulbasūtras* are known. They are *Śulbasūtras* of Baudhāyana, Āpastamba, Kātyāyana, Mānava, Maitrāyana, Vārāha, Vādhula, Māśaka, Hiranyakeśī. The names of the last two *Śulbas* are mentioned in the commentary of Karavindasvāmī on the *Āpastamba-Śulbasūtra* (11.11 and 6.10). But the copies of these two Mss. are still unavailable.
2. Datta, B. *The Science of Śulba*, University of Calcutta, 1932.

(i) to clean and level a plot on which the altar is to be constructed, (ii) to fix the east-west line (*prācī*), for .it was considered as the reference line, (iii) to construct bricks of specific sizes, (iv) to draw diagrams of specific sizes on the level ground on which the altar is to be constructed, (v) to lay bricks subject to certain principles. To perform these duties many specialists arose to solve constructional problems which required good deal of knowledge of geometry and mathematics. But there cannot be any doubt that the item (iv) together with item (v) helped much the development and nourishment of geometry. In the table (p. 106) an account is given of names of altars, types of altars, horizontal section of the altars (i. e., diagram on the ground) and sectional area (i. e., area covered by these diagrams.).

Instruments, Units and Bricks in the Śulbasūtras

Several instruments like *śaṅku*, bamboo rod, rope etc. and units like *aṅgula*, *pada*, *aratni*, *vyāyāma* etc. and bricks of definite shape were used by the *śulbakāras* for the construction of sacrificial altars. That these helped really in the construction of geometrical figures is explained here.

Instruments :

The following instruments were used in connection with altar construction.

i) *Śaṅku* It is a special type of pole. It was the main tool in a sunny day for ascertaining the cardinal directions i.e. the east-west and the north-south lines (Continued to p. 107).

Table—Name of Altars and other details

Names of Altars	Type	Horizontal Sections	Sectional area
1) *Āhavanīya*	Class I	Square	1 sq. *vyāyāma*
2) *Gārhapatya*		Circle or Square	,,
3) *Dakṣiṇāgni*		Semi-circle	,,
1) *Mahāvedi* or *Saumiki vedi*	Class II	Isosceles trapezium	972 sq. *padas*
2) *Sautrāmaṇi vedi*		Isosceles trap.	One third of the *Mahāvedi* i.e. 324sq. *padas*
3) *Paitṛkī-vedi*		Isosceles trap.	One ninth of the *Sautrāmaṇikī vedi*
4) *Prāgvaṃśa*		Rectangle	—
1) *Caturasraśyenacit*	ClassIII	Bird with square body (*ātman*) with square wings and square tail	$7\frac{1}{2}$ sq. *puruṣas*
2) *Vakrapakṣa vyastapucchaśyena*		Bird with bent wings and out—spread tail	,,
3) *Kaṅkacit*		Bird	,,
4) *Alajaci t*		Bird	,,
5) *Prauga*		Triangle	,,
6 (*Ubhayata prauga*		Rhombus	,,
7) *Rathacakracit*		Circle	,,
8) *Droṇacit*		Trough	,,
9) *Śmaśanācit*		Isosceles trap.	,,
10) *Kūrmacit*		Tortoise	,,

Mainly pole was fixed on plain uniform ground and a circle was drawn on the ground around the foot of the pole as centre. Then the line joining the points on the circle where the shadows of the pole before noon and afternoon cut the circle served to indicate the east-west line[1]. The actual east-west line might have been determined by the shadow of the pole on the equinox day and verified by the rising and setting points of the star *Kṛttikā*[2].

The *Śulbasūtras* have made use of the east-west line for the purpose of all constructions, and drawing works namely the construction of sacrificial altars. Haug has given at the end of his *Aitareya Brāhmaṇa*[3] a tolerably good map of the sacrificial compound of which the central line is an east-west line. In Vedas also, the measurements of the altars are given with reference to this central east-west line. The *Āhavanīya* fire is ordained by a Vedic text to be directly to the east of the *Garhapatya* fire. The fire *Dakṣiṇāgni* is to the south of this central east-west line. The *Pāśukī*-vedi for the Paśubandha sacrifice, a *Sautrāmaṇi* vedi which is called the *Mahāvedi* for the Somasacrifice; in all these vedis, the central east-west line remains fixed. In a *Saptapuruṣa*-

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1. *Kātyāyanaśūlbasūtra*, ch. 1, v. 2–3; *Sūryasiddhānta*, ch. 3, v. 1–51; *Brāhmasphuṭasiddhānta* of Brahmagupta, ch. 19, v. 1–20; *Siddhāntaśekhara* of Śrīpati, ch. 4, v. 14–16; *Līlāvatī* of Bhāskarācārya, ch. 11, v. 1–10; *Siddhāntaśiromaṇi* of Bhāskarācārya, ch. 7, v. 36–39; etc.
2. Apte, R. N. 'Constructive geometry of Vedic altars', *Annals of Bhandarkar Oriental Research Institute*, 7, pt. 1, p. 1–16, 1926.
3. *Śatapatha Brāhmaṇa*, 1.7.3. 22–27

cayana, for example, say, *Śyena-cayana* (bird-shaped construction), the whole figure is symmetrical with respeet to the central east-west line. Its wings and tail are symmetrically arranged. It is in fact the most important line and the whole sacrificial theatre, is symmetrical with respect to this central line. The central east-west line was called *pṛṣṭhyā* or *prācī*. In the construction of the altars in the *Śulbasūtras* the tern *prācī* was in fact a line of specific length drawn parallel to or on this central east-west line and which was drawn in such a way that the figure concerned must be symmetrical which the *prācī*.

In this connection, it is noteworthy that nothing is said about right angled triangle in the *Śulba sutra*. But the lines *pārśvamānī* (another line drawn parallel to *prācī* forming the limb of a specific figure) and *tiryaṅmānī* (a line drawn perpendicular to *prācī* as well as to *pārśvamānī* forming the limb of a specific figure) when drawn to represent the limbs of the same specific figure, it will obviously lead to the construction of a right-angle or a figure containing a right-angle. This seems to suggest that Baudhāyana and the posterior had the idea of a rightangle and right-angled figures.

The *tiryaṅmānī* was obtained by pulling out the middle point of a rope to the right or left of the *prācī* or *pārśvamānī* with its ends tied to two points on the *prācī* or *pārśvamānī*. This gives rise to an isosceles triangle where the equal sides are represented by the two halves of the ropes. The line joining the apex of this triangle i. e. the middle point of the rope, with the middle point of two fixed points on the *prācī* gives the line *tiryaṅmānī* which is obviously the perpendicular line.

ii) *Bamboo rod :* (a) *An instrument of measure*–The use of bamboo rod was traced as early as *Taittirīya Saṃhitā* (5.2.5.1.) and reappeared also in other *Saṃhitās* and in the *Śatapathabrāhmaṇa.* In the *Śrautas* and the *Śulbas*, its use considerably decreased.

(b) *Geometrical compass* – In one instance, the instrumental device given by Āpastamba (*Śulba* 9.1.) can be compared with modern geometrical compass in a limited sense. Three holes were made in a bamboo rod of desired length, two of these were made at its two ends and the third at the middle. By attaching the bamboo rod through a hole at its end or at the middle to a peg fixed on a ground and by moving its free end or ends, circles of two different radii can be constructed.

iii) *peg*, a short round piece of wood.

iv) *Rope* : In the *Śulba*, it has been largely used as a geometrical instrument. In Sanskrit, the words *dari*, *rajju*, *śulba*[1] and *sūtra* have indentical meanings. It means ordinarily 'a rope' or 'a cord'. *Rajju* as a measuring tape had been in use for a long time. In the *Vedas*, the word *dari* was used to mean a rope[2]. The term *śulba* does not occur in the Vedas. The word *rajju* appeared in the *Śatapathabrāhmaṇa*. In the *Śulbas*, both the terms *rajju* and *śulba*[3] are found to occur.

1. The word *Śulba* is derived from the term *śulb* meaning 'to measure'.
2. *Ṛg. veda* (1.162.5; 10.100.12); *Taittirīya Saṃhitā* (2.5.1.7); *Atharvaveda* (3.11.8; 6.121.2).
3. *Śatapatha Brāhmaṇa*, 10.2.3.11.

The term *rajju* also appears in later works.[1] Hence it is very difficult to determine when a cord as a unit of measure was introduced by the Indians.

The cord was prepared from *śana* (a kind of hemp, *Cannabis Sativa* or *Crotolaria Vuncea*), *balvaja* (made of grass, *Eleusine Indica*) or thin strip of bamboo (*veṇu*). It is also prepared from *muñja* (sedge like grass, *Saccharum muñja*), *kuśa* (sacred grass used at religious ceremonies).

The cord before use was cautiously checked. The cord which was not flexible, had no knots, and which was thin, uniform, soft and smooth was used A cord usually longer than the required size was used[2] in the *Śulba* for measuring line or surface. The land measuring or anything tied, was a common practice in almost all the countries of the world. The terms *akṣṇayā-rajju* (diagonal cord), *karaṇi*, *tatkaraṇi pārśva-mānī*, *tiryaṅmānī*, etc., clearly indicate lines drawn in

1. Kauṭilya mentions a measure of *rajju* which was nearly 60ft. (vide *Arthaśāstra* of Kauṭilya, Ed. by R. Shyama Sastri, 2nd ed. Mysore 1919, p. 107). In the script of Aśoka, the words *rajjuka* and *lajjuka* have been used. The words *rajjuka* and *lajjuka* are identical because in Sanskrit grammar, the letter 'r' can be used in place of 'l'. In the *Sthānāṅgasūtra*; a Jaina work, one of the ten sections of the members of mathematics is *rajju* (Sūtra, 338 and 747).
2. *Kātyāyauaśulba*, 7.13–14. These plants and materials are used for purpose of rope even today. There are various other materials and plants used for the purpose in British India. E. Balfour has given a list of about 156 principal cordage plants in British India (Balfour, E. *Cyclopaedia of India*, Second Edition, 4, 1873).

different directions.[1] Since most of the measurements were done by the help of a *rajju*, and surface measurements were in common among the *śulbakāras*, it is quite probable that *rajju* had its use in land surveying. Two different methods have been described for measurement with cords as an instrument, (1) method of measuring by one cord (*ekarajjuvidhi*), and (2) method of measuring by two cords (*dvirajjuvidhi*).

1) *Ekarajjuvidhi* was given by Āpastamba[1] as such. Having joined a cord of 18 *prakramas* (= BC) to another of 36 *prakramas*, (= AB), marks are to be given at 12 and 15 *prakramas* (CD = 12, CG = 15) from the free end of the smaller cord (Fig. 1). The two free ends of the combined cord are then fixed on a *prācī* line (= EF), the distance between which is that of the longer cord i.e. 36 *prakramas*, the free end of the longer cord being tied to the peg at the eastern end (E) and that of the shorter cord to the peg at the western end (F). Holding the combined cord at its mark (G) at 15, pull towards south as far as it goes and fix it there (G′) by means of a pole. The part of the cord of length 15 *prakramas* now becomes the line of *tiryaṅmānī* (FG′). Such marks (G′, Q′, Q′ and G″) on the ground, which fall on the *tiryaṅmānī* when the cord is fully stretched, called *nirañchana* marks. The same operation is repeated in the north. Obviously it gives the same result. Identical results are obtained if the free ends of the combined cord are interchanged. When the four points on the *tiryaṅmānī* thus obtained and

1. Vide list of teachnical terms.

2. *Āpastambaśulbasūtra*, 5. 2.

joined, it gives the figure of an isosceles trapezium (D″G″G′D′). This is the method of construction of an altar of definite shape with lengths of two sides of its face being 24 and 30 *prakramas* and the perpendicular distance between them being 36 *prakramas*.

It is evident from the Fig. 1 that this gives rise to a right–angled triangle in which $EF^2 + FG'^2 = EG'^2$. The construction of altar of the shape of square, rectangle, triangle and many other isosceles trapeziums by using this method have also been described in the *Śulba*[1].

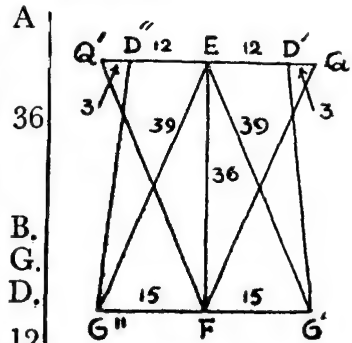

Fig. 1

The selection of a *nirañchana* mark by the *śulbakāras* shows that they had the knowledge of the properties of the right-angled triangle.

2) *Dvirajjuvidhi*[2]–In the construction of altars of various geometrical shapes by these methods, instead of a combined cord consisting of two parts (*ekarajju-vidhi*), two separate cords of different lengths (*dvirajjuvidhi*) were employed. Two ends of each of these cords were tied to the two poles on a *prācī* whose distance was less than that of the cord. A mark was made on each of these cords at a spot representing the *nirañchana* mark and putting the cord at the mark on each side of the *prācī*, two right angled-triangles could be obtained as in the case of combined cord By

1. *Āpastambaśulbasūtra*, 1.2, 5. 1–2, 6. 3–4, 6. 6–7, 7.1–2.
2. *Āpastambaśulbasūtra*, 5. 3–5.

joining the vertices of four such triangles, drawn with the help of these two cords, an isosceles trapezium could be constructed. Isosceless trapezium of different sizes were drawn by using the cords of multiple lengths with their corresponding *nirañchana* mark.

v) *Sphya :* A wooden rod cone-shaped at one end for drawing lines on the ground

Besides these instruments, rods were represented by the units of lengths namely *prādeśa*, *aratni*, *śamyā*, *vyāma*, *vyāyāma*, *puruṣa* etc. (vide table for unit of length given below) also used as measuring instruments.

Units :

The measure of the dimension of an object can be obtained by comparison with that of another object of the same species taken as the standard unit. In the *Śulbasūtras*, we come across linear, surface and volume units in connection with altar construction. *Śulbakāras* did not use separate names of the units for linear, surface and volume units. Surface and volume were expressed in terms of linear units. A table of linear units as found in the *Śulbasūtras*[1] is given below :

1 *aṅgula* = 24 *aṇus* = 34 *tilas* placed side by side.
1 *kṣudrapada* = 10 *aṅgulas*.
1 *pada* = 15 *aṅgulas*.
1 *prakrama* = 2 *padas* = 30 *aṅgulas*.
1 *aratni* = 2 *prādeśas* = 24 *aṅgulas*.
1 *puruṣa* = 1 *vyāma* = 5 *aratnis* = 120 *aṅgulas*.
1 *vyāyāma* = 4 *aratnis* = 96 *aṅgulas*.

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1. *Baudhāyanaśulba* (= BS), ch. 1, 3-21.

1 *pṛthā* = 13 *aṅgulas.*
1 *bāhu* = 36 *aṅgulas.*
1 *jānu* = 30 or 32 *aṅgulas.*
1 *iṣā* = 108 *aṅgulas.*
1 *akṣa* = 104 *aṅgulas.*
1 *yuga* = 88 *aṅgulas.*
1 *śamyā* = 36 *aṅgulas.*
1 *aṅgula* = $\frac{3}{4}$ *inch* (approximately).

As regards surface-measure *Baudhāyana* says that *Mahāvedi* measures 972 (sq.) *padas*[1]. The *Sautrāmanikī* vedi is 324 (sq) *padas.*[2] The corresponding references have unmistakably proved that the units denote square units and are meant for surface measure. This has been verified by actual calculation. Many references can be pointed from *the Śulbasūtras* in favour of this.[3]

As regards volume-measure, *Śulbakāras* were not very clear enough, since question of volume was not considered important. In the construction of altars, total height of the layers i.e. practically the height of bricks of altars remains constant for a specific construction.

Bricks :

The bricks of following shapes were used for the construction of altars, a table of which is given below.

1. *aṣṭaviṃśatyo'naṃ padasahasraṃ mahāvedi* (AS. 5. 7).
2. *trīṇi caturviṃśāni padaśatāni sautrāmaṇikī vediḥ* (AS. 5.9).
3. *vyāyāmamātrī gārhapatyaciti...........caturasretyekeṣām* .. ..
(BS. 2. 61-62);
śamyāmātrī catuḥsrakti ...sacaturasraviśeṣāt (BS. 2. 79-80);
paitṛkyā ṃdvipurṣāṃ samacaturasrāṃ kṛtvā...(AS. 2. 6), etc.

No.	Name of bricks	Shape	Dimension	Reference
1	*Caturthi* or *Caturbhāgiya* or *Aṇukā*	Square	30 *aṅg.*, 30 *aṅg.*	*BS. 3.11; 3.39 *AS. 11.3; 11.5; 11.7; 16.5; 14.7; 20.3.
2	*Ardhyā* (half of *Caturthi*)	(1) Triangle	30 *aṅg.*, 30 *aṅg*, $\sqrt{1800}$ *aṅg.*	BS. 3.63–64.
		(2) Rectangle	30 *aṅg.*, 15 *aṅg.*	....
		(3) Pentagon (*Haṃsamukhi*)	$7\frac{1}{2}$ *aṅg.*, 15 *aṅg*, $7\frac{1}{2}$ *aṅg.*, $\sqrt{450}$ *aṅg.*, $\sqrt{450}$ *aṅg.*	BS. 3.68.
3	*Pādyā* (one-fourth of *Caturthi*)	(1) Triangle	30 *aṅg.*, $\frac{\sqrt{1800}}{2}$ *aṅg.*, $\frac{\sqrt{1800}}{2}$ *aṅg.*	....
		(2) Four-sided	$22\frac{1}{2}$ *aṅg.*, 15 *aṅg.*, $7\frac{1}{2}$ *aṅg.*, $\sqrt{225}$ *aṅg.*	BS. 3.67.
		(3) Square	15 *aṅg.*, 15 *aṅg.*	....
4	*Adhyardhyā* (one and half of *Chaturthi*)	Rectangle	45 *aṅg.*, 30 *aṅg.*	BS. 3.287.
5	*Aṣṭami* (one-eight of *Caturthi*)	(1) Triangle	15 *aṅg.*, 15 *aṅg.*, $\sqrt{450}$ *aṅg.*	
		(2) Rectangle	15 *aṅg.*, $7\frac{1}{2}$ *aṅg.*	

*BS = *Baudhāyanaśulba* (Thibaut's Ed.). AS = *Āpastambaśulba* (Bürk's Ed.).

1. It is called *Caturthi*, as its sides are $\frac{1}{4}$th of a *puruṣa* (= 120 *aṅgulas*). *Puruṣa* was considered as the standard unit. It represents the height of a man measured by his own finger.

B. *Pañcamī brick*[1] *and its subdivisions :*

No.	Name of bricks	Shape	Dimension	Reference
6	*Pañcamī* (side one-fifth of a *puruṣa*)	Square	24 *aṅg.*, 24 *aṅg.*	BS. 3.11; 3.42. AS. 9.6.
7	*Ardhyā* (half of *Pañcamī*)	(1) Triangle	24 *aṅg.*, 24 *aṅg.*, $\sqrt{1152}$ *aṅg.*	BS. 3.120.
		(2) Rectangle	24 *aṅg.*, 12 *aṅg.*	BS. 3.44. AS. 9.6.
8	*Pādyā* (one-fourth of *Pañcamī*)	(1) Triangle	24 *aṅg.*, $\frac{\sqrt{1152}}{2}$ *aṅg.*, $\frac{\sqrt{1152}}{2}$ *aṅg.*	BS. 3.120.
		(2) Four-sided	18 *aṅg.*, 12 *aṅg.*, 6 *aṅg.*, $\sqrt{228}$ *aṅg.*	BS. 3.149.
		(3) Square	12 *aṅg.*, 12 *aṅg.*	AS. 9.1.
9	*Ādhyardhyā* (one and half of *Pañcamī*)	Rectangle	36 *aṅg.*, 24 *aṅg.*	BS. 3.43.
10	*Aṣṭamī* (one-eight of *Pañcamī*)	(1) Triangle	12 *aṅg*, 12 *aṅg.*, $\frac{\sqrt{1152}}{2}$ *aṅg.*	
		(2) Rectangle	12 *aṅg.*, 6 *aṅg.*	

1. It is called *Pañcamī* as its sides are $\frac{1}{5}$th of a *puruṣa* (= 12 *aṅgulas*).

C. *Adhyardhyā of a Pañcamī and its subdivisions* :

11	*Adhyardhyā*	Rectangle	36 *aṅg.*, 24 *aṅg.*	BS. 3.43, AS. 9.6.
12	*Ardhyā*	Triangle	20 *aṅg.*, 24 *aṅg.* $\sqrt{1476}$ *aṅg.*	BS. 3.121.
13	(1) *Dirghapadyā*	Triangle	36 *aṅg.*, $\frac{\sqrt{1872}}{2}$ *aṅg.*, $\sqrt{\frac{1872}{2}}$ *aṅg.*	B.S. 3.121
	(2) *Śulapādyā*	Triangle	24 *aṅg.*, $\frac{\sqrt{1872}}{2}$ *aṅg.*, $\frac{\sqrt{1872}}{2}$ *aṅg.*	BS. 3.121.
14	*Aṣṭamī* (*dīrghapādyā*)	Right-angled triangle	18 *aṅg.*, 12 *aṅg.*, $\frac{\sqrt{1872}}{2}$ *aṅg.*	BS. 3.122.
15	*Aṣṭamī* (*śulapādyā*)	Right-angled triangle	12 *aṅg.*, 12 *aṅg.*, $\frac{\sqrt{1872}}{2}$ *aṅg.*	BS. 3.122.
16	*Udhayī*	Triangle		BS. 3.122

When a square brick (*Caturthī* or *Pañcamī* is cut by one of its two diagonals each of the two equal parts gives rise to a brick [Nos. 2(1) and 7(1)] having the face of an isosceles triangle (Fig. 2). When it is cut by its two diagonals, each of the four equal pieces gives rise to a brick [Nos. 3(1) and 8(1)] having the shape also of an isosceles triangle (Fig 3), and when cut by its two bisectors of the sides, each of its four equal pieces gives rise to a brick [Nos. 3(3) and 8(3) having the face of a smaller square (Fig. 4). When divided simultaneously by the two diagonals and two bisectors of the sides, each of the eight equal pieces gives rise another brick [Nos.5(1) and 10 (1)] with the shape of an isosceles right-angled triangle (Fig 5). When any of the smaller square (Fig. 4) is divided into two equal halves by a bisector on one of the side, each one gives rise to a brick [Nos. 5(2) and 10(2) which is one-eighth of the original *Caturthī* or *Pañcamī* (vide Fig. 6).

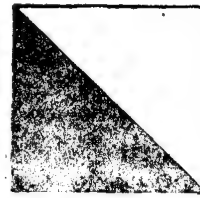

Fig. 2

Fig. 3

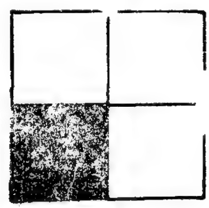

Fig. 4

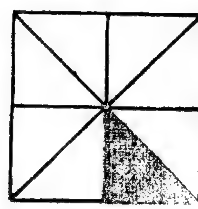

Fig. 5

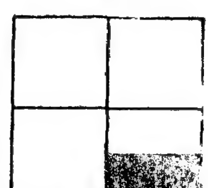

Fig. 6

When the longer side of the rectangular *Aṣṭamī* (Fig 6) is joined with one of the side of the isosceles right-angled triangle (Fig. 5), it gives rise to a brick [Nos. 3(2) and 8(2)] having shape of a trapezium (Fig. 7).

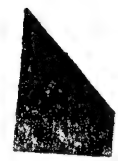

Fig. 7

When two such trapeziums are joined together along its longer sides, it gives rise to a *Haṃsamūkhī* brick [No. 2(3)] with its face representing pentagon (Fig. 8).

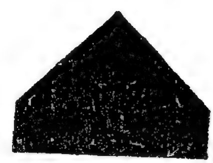

Fig. 8

When an *Adhyardhā* brick (one and half times of a square i.e. a rectangular brick) is cut along its two diagonals, it gives rise four triangular pieces two equal larger known as *Dīrghapādyā* bricks [No. 13(1)] and two equal smaller known as *Śulapādyā* bricks [No. 13(2)]. When a *Dīrghapādyā* having a triangular face ABC (See Fig. 9) is cut into two equal halves by cutting along the perpendicular line from the vertex A to the base of its tringular face, it gives a brick with the face of a right-angled triangle ABC(Fig. 9). Similarly when a *Śulapādyā* brick having a triangular face ACD (Fig 9) is divided into two equal halves by cutting it along the perpendicular line from the vertex A to the base of its triangular face, each gives a brick having a face of

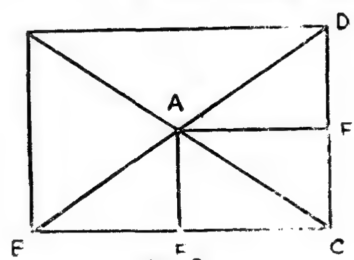

Fig. 9

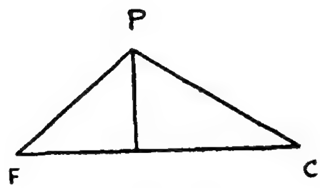

Fig. 10

the shape of a right-angled triangle A F D. If a brick having the face of a shape of a right-angled triangle derived from *Dirghpaādyā* or *Śulapādyā* (brick No. 14 or 15) be joined to a similar one derived from the *Pañcamī* along the sides of equal length [vide No. 10(1)] it gives rise to a brick [No. 16] whose face is represented by a scalene triangle PFC (Fig. 10).

From the consideration of the above description for the preparation of various bricks for the construction of altars, it can be concluded that the ancient Indians had the knowledge of different forms of geometrical figures viz., square, rectangle, isosceles triangle, isosceles right angled triangle, 'scalene triangle, trapezium, pentagon etc On the basis of these ideas, it is easy to see how the further development of geometrical knowledge was made in the *Śulbasūtras.*

Geometrical Propositions

Since the areas in classes I, II and III are fixed (vide p. 106) but the diagrams are different, various geometrical propositions undoubtedly evolved.

The following geometrical operations are found to have been studied in connection with different constructions :

i) Construction of a square on a given straight line.

ii) Construction of a circle equal to a given square and vice versa.

iii) (a) Doubling a circle.

(b) Geometrical method to find the value of $\sqrt{2}$ which is expressed as $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$ (approx.).

iv) (a) A theorem on the square of the diagonal which runs thus : "the diagonal of a square produces an area double of the original square".

(b) A general statement on the square of the diagonal of a rectangle now known as Pythagoras theorem runs thus : "the diagonal of a rectangle produces both (areas) which its length and breadth produce separately."

(v) Construction of an isosceles trapezium whose face, base and altitude are given.

(vi) Construction of a rectangle having given sides.

(vii) Construction of an isosceles trapezium whose area will be equal to a simple multiple or submultiple of, and which will be similar to, another isosce'es trapezium.

(viii) (a) Construction of a square equal to the sum of two unequal squares (*samāsavidhi*).

(b) Construction of a square equal to the difference of two unequal squares (*nirhāravidhi*).

(ix) Construction of a square equal to a simple multiple (or submultiple) of another square.

(x) Transformation of a rectangle into a square and vice versa.

(xi) Transformation of a square or a rectangle into an isosceles trapezium which shall have a given face.

(xii) (a) Transformation of a square or a rectangle into a triangle.

(b) Transformation of a square or a rectangle into a rhombus and vice versa.

(xiii) Transformation of an isosceles triangle into a square.

(xiv) Determination of area of a triangle ($= \frac{1}{2} \times$ base $\times$ altitude), trapezium (= base $\times$ altitude) and isosceles trapezium = [$\frac{1}{2}$ (face + base) $\times$ altitude].

There cannot be any doubt that a little thought about the construction of the altars in class I will presuppose the knowledge from (i) to (iv) and class II a knowledge from (v) to (vii) and class III from that of (viii) to (ix) and knowledge of these three classes together leads to the formulation of rules from (x) to (xiii) besides many others which are not discussed by the *Śulbakāras*. Though these geometrical results mainly occur in the *Śulbasūtras* of Baudhāyana, Āpastamba, Kātyāyana and others, the authors never claimed that they were the first to discover the principles of geometry or apply them to the problems of the construction of altars. Whenever they described a proposition, they referred to their previous workers as quoted here : (i) it is so recognised or prescribed by the authorities, (ii) thus they teach, (iii) it has been so said (*iti abhyupadiśanti iti vijñāyate, iti uktam* etc.). It might be pointed out here that some of the lines occurring in the *Śulbasūtras* are found to be literal quotations from the *Taittirīya Saṃhitā* or *Taittirīya Brāhmaṇa* or the corresponding *Āraṇyaka*. *Baudhāyana* is occasionally more explicit about his sources. In connection with certain differences of opinions amongst the altar builders about the proper size and shape of a particular altar, *Baudhāyana* is found appealing to the authorities of the *Brahmaṇa*, by name for the purpose of arriving at a satisfactory settlement.[1]

Theorem of Square on the Diagonal

The theorem is enunciated by Baudhāyana (c. 600 B. C.) as follows :

dīrghacaturaśrasyākṣṇayārajjuḥ pārśvamānī tiryaṅmānī ca yatpṛthagbhūte kurutastadubhayaṃ karoti |

"The diagonal of a rectangle produces both (areas) which its length and breadth produce separately". That is, in a rectangle ABCD, $AC^2 = AB^2 + BC^2$. This is a most general statement and was enunciated first by Baudhāyana. The proposition is stated in almost identical terms also by Āpastamba and Kātyāyana[2]. Baudhāyana further says that the theorem is easily verified from the following relations :

$$3^2 + 4^2 = 5^2$$
$$12^2 + 5^2 = 13^2$$
$$15^2 + 8^2 = 17^2$$
$$7^2 + 24^2 = 25^2$$
$$12^2 + 35^2 = 37^2$$
$$15^2 + 36^2 = 39^2$$

No proof of this theorem is given by Baudhāyana and other Śulba writers, since it is beyond their tradi-

1. *BaudhāyanaŚulbasūtra*, 2. 15-19; 2. 35.
2. Datta, B. — *The Science of Śulba*, p. 104-132, Calcutta University, 1932.

tions. Zeuthen[1], Cantor[2], Vogt[3], Cajori[4], and Heath[5] have expressed the view that the general statement was possibly the result of an induction from a small number of cases of right-angled triangles having sides in rational numbers known to them. But this is not the actual case. Our discussion on rational rectangles and construction of geometrical figures amply justify that the general character of the theorem was rightly understood by the Sulbakāras. This has been discussed by Burk[6], who concluded that the theorem of square on the diagonal was known and proved in all its generality by the Indians long before the date of Pythāgoras (c-540 B. C). He went further that so far from Indian Geometry being indebted to the Greek, the much travelled Pythāgoras probably obtained his theory from India.[7] In this connection a number of

1. Zeuthen, H. G. 'Theorem de Pythagore Origine de la geometrie scientifique', *Comptes Rendus du IIme Congress internationale de Philosophie*, Geneve, 1904; 'Sur l' Arithmetique Geometrique des Grecs et des Indiens', *Bibliotheca Mathematika*, Series 3, 5, p. 97-112, 1904,
2. Cantor, M. 'Uber die alteste indische Mathematique,' *Archiv der Mathematik und Physik*, 8, p. 63-92, 1905.
3. Vogt, H. 'Haben die alten Inder den Pythagorischen Lehrsatz und das Irrationale Gekannt ?', *Bibliotheca Mathematica*, 7, pt. 3, p. 6-23, 1906.
4. Cajori, F. *A History of Mathematics*, 2nd revised and enlarged edition, p. 86, 1919.
5. Heath. T. L. *A History of Greek Mathematics*, *1*, p. 147, Oxford, 1921.
6. *Zeitschrift der deustschen morgenlandischen Gesellschaft* (*ZDMG*), *55*, p. 546.
7. *ZDMG*, *55*, p. 575; Jones, Sir William, *Works*, 8, p. 236;

conjectures by Bretschneider[1], Gow[2], Hankel[3], Allman[4], Heath[5], Dutta[6] and others as to the way the proof of the theorem could have been arrived at by the Indians are available.

The theorem of square on the diagonal of a rectangle i.e. $a^2+b^2=c^2$, is usually known as Pythagorian theorem and is universally associated with the name of the Greek Pythagoras (c. 540 B C.). In fact the relation $3^2+4^2=5^2$ and some such relations have been used by Pythagoras, and the evidence of any general statement regarding this is not yet available. Actual proof was first given by Euclid (c. 300 B. C.).[7] Proclus (c 460 A. D.), the commentator of Euclid remarked : "for my part, while I admire those who first observed the truth of the theorem, I marvel more at the writer of the Elements, not only because he made it first by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragable arguments of science in the sixth

Colebrooke, T. *Miscellaneous Essays*, *1*, p. 436; Garbe R. *Philosophy of Ancient India*, p. 39 etc.

1. Bretschneider, C. A. *Die Geometrie und die Geometer von Eukleides* ?, p. 82, Leipzing, 1870.
2. Gow, J. *A Short History of Greek Mathematics*, Cambridge, p. 155, 1884.
3. Hankel, H. *Zur Geschichte der Mathematik in altertum in mitelalter*, p. 98, Leipzig, 1874.
4. Allman, G. *Greek Geometry from Thales to Euclid*, Dublin, p. 37, 1889.
5. Heath, T. L. *The Thirteen Books of Euclid's Elements*, Cambridge, *1*, p. 352, 1908.
6. Datta, B. *The Science of Śulba*, p. 159, 1932.
7. *Elements*, Book 1, Proposition 47.

book.[1] Heath admitted with the remarks : "it is difficult for us to be more positive than Proclus was".[2] Hence it is far from conclusive that Pythagoras stated the above general theorem. Variety of evidence is at present available that the practical use of the theorem was current in old Babylonian times (c. 1800–1600 B. C.). The evidence for this is found in certain Babylonian cuneiform tablets. No general theorem was found to have been mentioned. It has been conclusively proved by Neugebauer that Pythagoras has derived his "Number theorem of Universe" as well as the so called Pythagorian Theorem from the Babylonian cuneiform tablets[3]. The Chinese knew a similar relation which appeared in *Chou Pei* (4th century B. C.) and the most of its credit is connected from the time of its first commentator Chao Chun Chhing (3rd century A. D.)[4]. In India, the proof first appeared in the work of Bhāskara II[5] (c. 1150 A. D.). Needham remarks "Bhāskara's treatment derives from the Chou Pei".[6] This is not however true, for the proofs of Bhāskara II and that stated to be of *Chou Pei* can readily

1. Proclus' commentary on the Euclid, Book I, is one of the two main sources of information as to the History of Greek Geometry, the other being the collection of Pappus.
2. Heath, T. L. *A Manual of Greek Mathematics*, Dover Publications, p. 96, 1963.
3. Neugebauer, O. *The Exact Sciences in Antiquity*, p. 28-42, 1952.
4. Needham, J. *Science and Civilization in China*, *3*, p. 95.
5. *Bījagaṇita* of Bhāskara II. *Vide* Sudhākara Dvivedi's edition, p. 70.
6. Needham, J. *Ibid.*, *3*, p. 19.

be deduced from a number of constructions described in the *śulbasūtras*[1].

Value of $\sqrt{2}$

The value of $\sqrt{2}$ given by Baudhāyana[2] in his *Śulbaśūtras* is :

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} \text{ (approx.).}$$

This is also given by Āpastamba[3] and Kātyāyana.[4] In terms of decimal fractions, this becomes $\sqrt{2}$ = 1.4142156. According to modern calculation $\sqrt{2}$ = 1.414213...... .

Thibaut, Rodet, Datta and others gave possible method of solution for arriving at the value as follows:–

i) *Thibaut's proof*[5].

We know $17^2 = 2.12^2 - 1$. Thibaut argued, by how much side 17 must be diminished in order that the square on it may be 2.12^2 exactly. Since $2 \times 17 \times \frac{1}{34} = 1$, he observed, two strips each of $\frac{1}{34}$ (approximately) are to be cut off from a square with 17 as side to obtain the square 2.12^2 (i.e. $12^2 + 12^2$)

$$\text{Hence } \left(17 - \frac{1}{34}\right)^2 = 2.12^2$$

$$\text{or } \frac{17 - \frac{1}{34}}{12} = \sqrt{2}$$

1. Datta, B. *The Science of Śulba*, p. 77, University of Calcutta, 1932.
2. *pramāṇaṃ tṛtīyena vardhayettacca caturthenātmacatustriṃśonena, saviśeṣa* (*Baudhāyana-śulba*, 1.61–2).
3. *Āpastambaśulba*, 1.6.
4. *Kātyāyanaśulba*, 2.13.
5. Thibaut, G. 'On the Śulbasūtras', *Journal of the Asiatic Society of Bengal*, 44, Pt. I, No. 3, p. 239–41, 1875.

Again, $17 - \frac{1}{34} = 12 + 4 + 1 - \frac{1}{34}$

or, $17 - \frac{1}{34} = 12\left(1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}\right)$

or, $\frac{1}{12}\left(17 - \frac{1}{34}\right) = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$

or, $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$

For Baudhāyana's selection of units of 12 *aṅgulas* (= one *pada*) and 34 *tilas* (= one *aṅgula*), Thibaut expressed no doubt for the selection of arbitrary relation $17^2 = 2.12^2$ (approx.) contributing to the origin of the formula.

ii) *Rodet's approximation*[1].

According to Rodet, the approximation of $\sqrt{2}$ adopted by *Śulbakāras* may be obtained by successive approximation.

$$\sqrt{a^2 + r} = a + \frac{r}{2a+1} + \frac{\frac{r}{2a+1}\left(1 - \frac{r}{2a+1}\right)}{2\left(a + \frac{r}{2a+1}\right)} + e$$

where e is the fourth term approximation. Rodet might have obtained as follows :

$$\sqrt{a^2 + r} = a + \frac{r}{2a+1} \text{[two terms approximation]}$$

where ($2a + 1$) is the difference of the squares of a and the next positive integer ($a + 1$).

For third term approximation, let

$$\sqrt{a^2 + r} = a + \frac{r}{2a+1} + e_1$$

$$= \frac{2a^2 + a + r}{2a+1} + e_1$$

1. Datta, B. *The Science of Śulba*, Calcutta University, 1932.

Squaring and neglecting e_1^2, we get

$$2\left(\frac{2a^2+a+r}{2a+1}\right).\quad e_1=(a^2+r)-\left(\frac{2a^2+a+r}{2a+1}\right)^2$$

$$=\frac{r(2a+1-r)}{(2a+1)^2}$$

$$\therefore e_1=\frac{r(2a+1-r)}{2(2a+1)(2a^2+a+r)}=\frac{r(2a+1-r)}{2(2a+1)^2\left(a+\frac{r}{2a+1}\right)}$$

$$=\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}$$

Like wise the fourth term approximation is obtained. Obviously following above we write,

$$\sqrt{2}=\sqrt{1^2+1}=1+\tfrac{1}{3}$$

let $\sqrt{2}=1+\frac{1}{3}+e_1=\frac{4}{3}+e_1$

Squaring both sides, and cancelling e_1^2 from both sides, we get

$$\frac{8}{3}e_1=2-\frac{16}{9}=\frac{2}{9}$$

$$\therefore e_1=\frac{2}{9}\times\frac{3}{8}=\frac{1}{12}=\frac{1}{3.4}$$

$$\therefore \sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}$$

Again let $\sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}+e_2$

$$=\frac{17}{12}+e_2$$

Squaring both sides and cancelling e_2^2 from both sides,

$$\frac{17}{6}e_2=2-\left(\frac{17}{12}\right)^2=-\frac{1}{144}$$

$$\therefore e_2 = -\frac{1}{144} \times \frac{6}{17} = -\frac{1}{12.34} = -\frac{1}{3.4.34}$$

$$\therefore \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} \text{ (approx.)}$$

The method described later by Gurjar[1] and Gupta[2] is same and is no improvement over that of Rodet.

iii) *Datta's Proof*:

Datta has, however, explained that the value of $\sqrt{2}$ was possibly obtained from a geometrical method.[3] This method consists in constructing a square with area equal to the sum of the areas of the two other squares having sides of one unit in length. For this, one of the two squares having side of unit length is divided into three equal parts by lines drawn parallel to one of its side. Each of these parts forms a rectangular piece of one unit in length and one third unit in width. Two of these rectangular parts are then joined lengthwise to the two adjacent sides of the other unit square. This leaves a square hole at one of the corners of the enlarged unit square. This square-hole will have a side of one-third unit in length. The remaining rectangular piece of the divided unit square is again subdivided into three equal parts each forming a square of side one-third unit length. One of these square is fitted into the square-hole formed at one of the corners of the enlarged squāre. Each of the remaining two squares

1. Gurjar, L. V. 'The Value of $\sqrt{2}$ given in the Sulbasūtras.' *Journal of the University of Bombay*, ns. 10, pt. 5, 6–10, 1942.
2. Gupta, R. C. 'Baudhāyana's value of $\sqrt{2}$, *The Mathematics Education*, 6, No. 3, 77–79, 1972.
3. Datta, B. *The Sciencc of Śulba*, p. 192-94, Calcutta University 1932.

is again subdivided into four equal rectangular pieces having length of $\frac{1}{3}$ unit and width of $\frac{1}{3.4}$ unit. Eight of these small rectangular slips are placed lengthwise side by side on the two adjacent sides of the enlarged square with four on each side. This again leaves a square-hole at the corner having a side of length $\frac{1}{3.4}$ unit. Now two equal thin slips have to be deducted from the two adjacent sides of the enlarged square under construction, the width of each of the slips is therefore given by

$$\frac{\left(\frac{1}{3.4}\right)^2}{2\times\left(1+\frac{1}{3}+\frac{1}{3.4}\right)}=\frac{1}{3.4.34}$$

Hence the side of the desired square is given approximately by $1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}$ which gives the value of $\sqrt{2}$. This method is simpler and straight forward than that of Thibaut. Datta also discussed fully the knowledge of value of

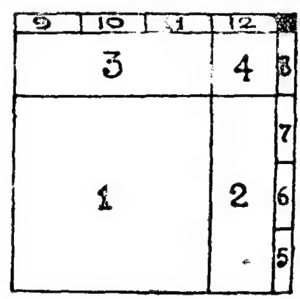

Fig. 11

the irrationality of $\sqrt{2}$ of Hindus and concluded that the irrationality of $\sqrt{2}$ was known to the ancient Hindus, though no direct method is available in the Śulba texts. Sen has also discussed this in A Concise History of Science in India (p. 154–55).

In Babylon, a small cunei-form tablet of the old Babylonian times (c. 1800–1600 B. C.) is now traced in the

Yale Babylonian collection (No. 7289) which shows a square with its two diagonals.[1] Three numbers in sexagesimal system are inscribed on it. These three numbers as represented by Neugebauer[2] as the value of diagonal, a side and the value of $\sqrt{2}$ (since $d = \sqrt{2}a$). Here $\sqrt{2} = 1,24,51,10$. Now Hindu[3] value $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$, if changed into sexagesimal unit, gives $\sqrt{2} = 1,24,51,10,36$....... This has led to the speculation that the Hindu system of calculating the value of $\sqrt{2}$ was developed from that of the Babylonians. Two points are important in this connection so far as the Hindu origin is concerned : i) In the period, there was no use of sexagesimal system, ii) the rule was clearly stated thus : "the side of a square is to be increased by its third, the third again by its fourth part less its (fourth parts) thirty-fourth part". We have no evidence of any such method of which the Babylonians were aware. As regards Greek sources, many approximations to the values of $\sqrt{2}$ were current but such a methood and correct approximation of $\sqrt{2}$ did not occur.[4]

1. Neugebauer, 0. *The Exact Science in Autiquity*, 1952 *Plate–6a.*

2. Neugebauer, 0. Ibid., p. 34.

3. $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$

$= 1 + \frac{5}{12} - \frac{1}{3.4.34}$

$= 1, 25 - 0; 0, 8, 49, 24$

$= 1, 24, 51, 10, 36$.......

4. Heath's translation of Euclid, 3, p. 2; Heath's *History of Greek Mathematics*, 1, p. 155.

Technical Terms in the Śulba sūtras

The technical geometrical terms found in different treatises of the *Śulbasūtras* of Baudhāyana, Āpastamba and Kātyāyana are tabulated below.

Terms in the *Śulbasūtra*	English Equivalents
1. *akṣa* or *akṣṇayā*	Diagonal.
2. *anta*	Point of intersection.
3. *bhūmi*	Area.
4. *caturasra* or *sama-caturasra*	Square.
i) *animata caturasra*	Isosceles trapezium.
ii) *dīrghacaturasra*	Rectangle.
5. *iṣu*	Arrow, altitude.
6. *karṇa*	Angle.
i) *eka–karṇa*	A geometrical figure of one angle.
ii) *dvi–karṇa*	A geometrical figure of two unequal angles.
iii) *pañca–karṇa*	Pentagon.
iv) *tri–karṇa*	Triangle.
7. *karaṇī*	(a) The side of rectilinear geometrical figure (Geometry). (b) Square root (Arithmetic).
i) *dvikaraṇī*	(a) Diagonal of a square. (b) $\sqrt{2}$
8. *kṣetra*	Plane figure.
9. *kṣetra–jñāna*	(Familiar with) the knowledge of the plane figures.

Terms in the *Śulbasūtras*	English Equivalents
10. *madhya*	Centre of a circle. This term is also used in more general sense for the middle most point of a square, or a rectangle.
11. *maṇḍala* or *parimaṇḍala*	Circle.
i) *vṛtta*	Circle.
12. *pariṇāha*	Circumference of a circle.
13. *pārśvamānī*	Longer side of a rectangle or side of a square which falls on the *prācī* or goes parallel to *prācī*.
14. *rekhā* or *lekhā*	Line.
15. *ṛju–lekhā*	Straight line.
16. *śakaṭamukha*	Isosceles triangle.
17. *saṃdhi*	Point of intersection.
18. *saviśeṣa*	(a) (Simply) a diagonal of a square. (b) Calculated value of a diagonal of a square. Symbolically, *saviśeṣa* of $a = a + \frac{a}{3} + \frac{a}{3.4} - \frac{a}{3.4.34}$. (For details, vide B. Datta' *The Science of Śulba*, p. 196–200.)

Terms in the *Śulbasūtra*	English Equivalents
19. *srakti*	Corner or angle.
i) *catuḥsrakti*	Square. The term is very old and is found in the *Taittirīya Saṃhitā*. 6. 6.10. 1; *Vājasaneya Saṃhitā*. 38.20; *Śatapatha Brāhmaṇa*. 2.6.1.36; Āpastambha *Śrautasūtra*. 7.5.1.
20. *tiryak*	The breadth of a rectangle or any side of a symmetrical figure which is perpendicular to *prācī*.
21. *tulya*	Equal.
22. *vakrāṅga*	Having curved face or elliptical.
23. *viṣama caturasra*	Quadrilateral.
24. *viṣkambha*	Diameter of a circle.
25. *vyāsa*	Diameter of a circle.

Śulbasūtras and the Later Mathematics

Certain scholars think that later Mathematics bears no noticeable relation with the *Śulbasūtras*. Kaye[1] pointed out that later mathematicians completely ignored the mathematical contents of the *Śulbasūtras*. He goes even further by saying that no Indian writer earlier than the 19th century is known to have referred

1. Kaye, G. R. *Indian Mathematics*, p. 3, Calcutta, 1918; Vide also p. 9.

to the *Śulbasūtras* as having any mathematical value. This has been supported by Cajori[1] and smith.[2]

The real fact has not been reproduced by these scholars. Before discussing the extent to which mathematics was cultivated after the *Śulba* period, some attempts must be made to understand the mental background of the people of that time. When the Vedic tradition was on the wane, a current of revolutionary thought began to interplay. Basically this current of thought was intellectual – an upsurge of a desire to solve the mental problems. By this time, Buddha and Mahāvīra appeared in forefront with their great personalitics Owing to the rise of Buddhism and Jainism which preached against the Vedic sacrifices for several centuries, the Vedic sacrifices were on the wane. As a result, the occasions for the construction of altars requiring high skill and ingenuity on the part of the constructors were few and far between. On the other hand, it should be observed that the Hindu geometry in the Śulba-period was of extremely practical nature and it was guided by religious motives whereas the later mathematicians were generally guided by academic interests. Nevertheless, it is not correct to say that *Śulbasūtras* became obsolete and were overlooked by all in India. The following items will readily show that the later mathematicians were not unaware of the *Śulbasūtras*.

1. Cajori, F. *A History of Mathematics*, 2nd revised and enlarged edition, p. 84–86.
2. Smith, D. E. *History of Mathematics*, 1, p. 145, Dover Publication.

From the table of the technical terms supppplied, one can easily see that the following terms of the *Śulbasūtras* were used almost in the same sense in the later mathematical works viz. *caturasra, karṇa, karaṇi, kṣetra, vṛtta, parināha, rekhā, viṣama–caturasra, viṣkambha, vyāsa*, etc. Āryabhaṭa I (c. 476 A. D.), used the terms *citighana, vargacitighana* and *ghanacitighana* for sum of triangular numbers, the sum of the squares of the natural numbers and the sum of the cubes of the natural numbers respectively. *Citi* means a pile or altar (of bricks) and *ghana* means cubic contents. These terms may be explained only if we assume that Āryabhaṭa I studied these series in relation to construction of altars. And this is not improbable, since we have shown that the mathematics first developed in India in connection with the construction of Vedic altars.

Various rules of construction and transformation occurring in the *Śulbasūtras* are : 1) the so–called Pyathagorian theorem, 2) the properties of similar figures, 3) the circle squaring, 4) construction of right triangle, 5) calculation of areas, through different division and transformation of figures. The first two of these topics enter largely in the later Hindu Mathematics and astronomy which will be seen in our subsequent discussion on geometry of Āryabhaṭa I, Brāhmagupta, Bhāskara II etc. That the later Hindu mathematicians did not lose their interest as regards circle squaring can be found from our discussion on the method of construction of a right triangle. It is mainly found in the works of Brahmagupta, Mahāvīra and others. The item 5) does not appear in the later

Hindu works on mathematics in the particular manner in which they occur in the *Śulbasūtras*. But that the later mathematicians were interested in finding the area of triangular, tetragonal and circular plots and the volume of a right prism, sphere, can also be seen from our discussion of geometry of Āryabhaṭa I, Brahmagupta, Mahāvīra and others. The history of development of arithmetic and algebra of the post-Vedic mathematics also shows that they were not unaware of the work of the *Śulbasūtras*. But in this connection, it must be clear that the first two items were used more significantly than others. Whitney's remark may be noted in this connection, "The two main principles, by aid of which the greater portion of all Hindu calculations are made, are, on one hand, the equality of the square of the hypotenuse in a right angled triangle to the sum of the square of the other two sides, and on the other hand, the proportional relation to the corresponding parts of similar triangles."[2]

The *Śulbas* continued to remain classics with the priestly class having academic interest and were followed in practice wherever, there were occasions during the succeeding centuries. That people interested in them did never disappear will be sufficiently realised from the necessity that arose in the middle ages from writing commentaries on the *Śulbasūtras*. Two commen-

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1. *Āryabhaṭīya* (*gaṇitapāda*), v. 21–22.
2. Vide Whitney's Note under *Sūryasiddhānta*, ch. 2, v. 27. Proportional relation to the corresponding parts of the similar triangles is not used in the *Śulbasūtras*, but it may be that this relation is a development to the relation used in the type of similar figures having different areas.

taries are now available on the *Śulbasūtras* of Baūdhayana. They are *Śulbadīpikā* of Dvārakānātha Yajvā (between 5th to 8th centuries), *Ṣulbamimāṃsā* of Vyaṅkaṭeśvara Diksita. On the *Śulbasūtra* of Āpastamba, five commentaries are available viz., *Vyākhyā* by Karavindasvāmī (between 5th to 8th century A. D.), *Śulbasūtra-bhāṣya* by Kapardisvāmī (before 12th century A. D.), *Vyākhyā* by Sundararāja (c. 1575 A. D.), *Āpastamba Śulba-bhāṣya* by Sundarasūri, *Vyākhyā* by Gopāla Yajvā. Similarly four commentaries are written by Gaṅgādhara, Karka, Mahīdhara and Rāma Candra on the *Kātyāyana-Śulbasūtra* etc. These commentaries, apart from interpretations, have collated many verses on the *Śulbasūtras* which show that these texts became popular in hater times to suit different purposes. For manuscripts and other references, our *Bibliography*[1] may be consulted.

Nature of Later Works

From 300 B. C. to 300 A. D. we get a picture of early Jaina geometry. No single Jaina treatise of this period dealing with geometry and mathematics in general is known to us. Though geometry was admired as "lotus of mathematics"[2], yet we get only scrapy references here and there. In the *Sthānāṅgasūtra*[3] (c. 300 B. C.), an early Jaina work, *rajju*, an instrument of measurement and *rāśi*, a solid quantity meaning

1. *A Bibliography of Sanskrit Works on Astronomy and Mathematics*, by S. N. Sen, A. K. Bag and S. R. Sarma, National Institute of Sciences of India, New Delhi, 1966.
2. *Sūtrakṛtāṅgasūtra*, 2nd Śrutaskanda, ch. 1, v. 154.
3. Sūtra 747.

measurement of solid bodies are the two topics mentioned under geometry[1]. Though the term *rajju* does not appear in later works, there will be no difficulty in recognising it as referring to plane geometry (*kṣetra*). It is synonymous with the term *Śulba* of the Vedic period. A commentator of the Jaina work, *Gaṇitasāra-saṃgraha* has rightly identified it (*Śulba*) with the *Kṣetragaṇita*. The term *rāśi* appears in later works where it means measurements of quantity of grains. In the later Hindu treatises, the topic on *rāsī* forms a part of *Khāta*[2] (cubic figures). Some technical terms of geometrical importance are used in the *Sūrjaprajñapati*, *Bhāgavatisūtra*, *Tattvārthādhigamasūtra* of Umāsvati (c. 150 B. C.) etc. Probably these Jaina authors had more detailed knowledge, but the treatises written by them are lost. Several relations regarding circumference of a circle and their diameter are available in the *Tattvārthādhigamasūtra*, *Kṣetrasamāsa*, *Trilokasāra* and *Gommaṭa-sāra* and other Jaina works.[3]

The *Āryabhaṭīya* (c. 499 A. D.) of Āryabhaṭa I consists of four sections of which, the section of *Gaṇita-pāda* (or mathematics) deals with geometry. It gives some formulae for calculating the area of a triangle, circle and trapezium and the volume of a sphere and pyramid. It also states the theorem of square on the diagonal. Brahmagupta (c. 628 A. D.)

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1. *Kṣetragaṇita* is the name for geometry appearing in the *Gaṇitasārasaṃgraha* (See ch. 7).
2. It deals with the content of the well whose length, breadth and depth are known.
3. Vide Preface of the *Gaṇitatilaka*, d E. by H. Kapadia.

treated the subject (*vide Brāhmasphuṭasiddhānta*, ch. 12) in the section on : (i) *Kṣetra* (plane figures), (ii) *Khāta* (cubic figures), (iii) *Citi* (piles of bricks–solid figures), (iv) *Krakaca* (saw–measurement of height of a solid like wood by cutting it into pieces with a saw) and (v) *Chāyā* (shadow–plane figures). Brahmagupta's rules consist mainly of certain formulae and solution of certain plain rectilinear figures such as triangles and quadrilaterals. The classification and terminology of the Jaina author Mahāvīra (c. 850 A. D.) are more accurate and they clearly express the relation of those subjects to the main body of the *Gaṇita*. He calls first the *Kṣetragaṇita* and rest by the name *Khātagaṇita* or simply *Khāta*. Śrīdhara (c. 900 A. D.) and Bhāskara II (c. 1150 A. D.) included *rāśi* (solid figures) in addition to topics discussed by Brahmagupta. In addition to some formulae, Bhāskara II gave a new proof to theorem of square on the diagonal. But on the whole, geometrical knowledge of these authors is mostly based on the consideration of practical needs. There were no difinitions, no postulates, no axioms. Thereafter no original works on mathematics discussing geometry came out. But in the *Gaṇitakaumudī* of *Nārāyaṇa* (c. 1356 A. D.) and *Pāṭīsāra* of Muniśvara (c. 1603 A. D.), We fiind some independent solutions of geometrical problems previously worked out by Brahmagupta and Bhāskara II, though a quite a large number of commentaries[1] were written on the famous

1. Vide *A Bibliography of Sanskrit Works on Astronomy and Mathematics*, Part I, by S. N. Sen, A. K. Bag and S. R. Sarma, National Institute of Science of India, New Delhi, 1966, p. 25.

work *Līlāvatī* by Bhāskara II during this period. Euclid's Elements came to be known to Indian scholars in the mid-fourteenth century though it was actually translated into Sanskrit first by Jagannātha (c. 1718 A. D.) (Vide p. 50–51).

Technical Terms[1] *in the Jaina and other Works*

(a) In the *Sūryaprajñapati* (c. 300 B. C.) the following terms are traced :—

Terms	English Equivalents according to Weber[2]
1. *cakrārdhacakravāla*	Semi-circle.
2. *chatrākāra*	Segment of a sphere.
3. *samacakravāla*	Circle.
4. *samacaturasra*	Square, rhombus, even square.
5. *samacatuṣkoṇa*	Square, rectangle, even parallelogram.
6. *viṣamacaturasra*	Quadrilateral, oblique square.
7. *viṣamacakravāla*	Ellipse.
8. *viṣamacatuṣkoṇa*	Quadrangle other than square, oblique parallelogram.

(b) In the *Bhagavatī-sūtra* (c. 300 B. C), we find the geometrical terms as follows :—

1. For technical terms under (a), (b) and (c), vide Kapadia's note in the Preface of *Gaṇitatilaka* of Śrīpati.
2. Indische Studien, **10**, p. 274.

(b) Technical terms in the *Āryabhaṭīya* (ch. on *Gaṇita*).

Technical terms in the *Āryabhaṭīya* (c. 499 A. D.) of *Āryabhaṭa I*	Modern English Equivalents
1. *bhūja* (v. 15, 17)	Base of a right angled triangle.
2. *dvādaśaśri* (v. 3, 4)	Cube.
3. *caturbhūja* (v. 13)	Quadrilateral.
4. *ghana* (v. 6)	Volume.
5. *gola–phala* (v. 17)	Volume of a sphere.
6. *karṇa* (v. 13, 17)	(i) Hypotenuse of a right angled triangle. (ii) Diagonal of a square o a rectangle.
7. *koṭi* (v. 16, 17)	Perpendicular of a right angled triangle.
8. *parināha* (v. 7, 10)	Circumference.
9. *phala* (v. 6)	Area.
10. (*ghana*) *ṣaḍaśri* (v. 6)	Triangular pyramid.
11. *samadala–koṭi* (v. 6)	A perpendicular which bisects the base of a triangle (refers to isosceles triangle).
12. *tribhūja* (v. 6, 13)	Triangle.
13. *viṣkambha* (v. 7, 10)	Diameter.
14. *vṛtta* (v. 10, 13, 17)	Circle.
15. *vṛtta–phala* (v. 17)	Volume of a sphere.

Terms	English Equivalents
1. *āyata*	Rectangle.
2. *caturasra*	Quadrilateral.
3. *ghana*	Solid.
4. *ghana-tryasra*	Triangular pyramid.
5. *ghana caturasra*	Cube.
6. *ghana-āyata*	Rectangular parallelepiped.
7. *ghana-vṛatta*	Sphere.
8. *ghana-parimaṇḍala*	Elliptic cylinder.
9. *parimaṇḍala*	Ellipse, circular or curved figure.
10. *pratara*	Plane.
11. *tryasra*	Triangle.
12. *valaya-vṛtta*	Circular annulus.
13. *valaya-tryasra*	Triangular annulus.
14. *valaya-caturasra*	Quadrangular annulus.
15. *vṛtta*	Circle.

(c) In the *Tattvārthādhigamasūtra* of Umāsvati (c. 150 B. C), are found the following terms :

Terms	Modern English Equivalents
1. *dhanuṣkāṭha*	Arc of a circle less than a semidiameter.
2. *iṣu*	Arrow or perpendicular from the vertex to the base of a triangle.
3. *jyā*	Chord.
4. *viṣkambha*	Diameter.
5. *vṛttaparikṣepa*	Circumference.

Terms	English Equivalents
15. *koti* (v. 24, 35)	Perpendicular of a right angled triangle or upright.
16. *kṣetra* (v. 34, 44)	Plane figure.
17. *mūla* (v. 24, 26)	Square root.
18. *pada* (v. 23)	Square root.
19. *śara* (v. 33,41,42 etc.)	Arrow.
20. *tribhūja* (v. 27, 31)	Triangle.
(i) *dvisama tribhūja* (v. 33)	Isosceles triangle
(ii) *trisamabhūja* (v. 37)	Equilateral triangle.
(iii) *viṣama tribhūja* (v. 29)	Scalene triangle.
21. *viśeṣa* (v. 92)	Difference.
22. *viṣkambha* (v. 27)	Diameter.
23. *vṛtta* (v. 41)	Circle.
(i) *vahir-vṛtta* (v. 27)	Exterior circle.
24. *vyāsa* (v. 41,42,43 etc.)	Diameter.

Basis of Nomenclature of the Geometrical Figures.

There are two fundamental bases usually adopted for naming geometrical figures depending on the relation between i) sides constituting the figure and ii) angles contained in the figure. Shortly they may be called side basis and angle basis. In Indian treatises both these types are found to occur. As already stated in the list for technical terms of the *Śulbasūtras*, the following terms are found to occur : *same-caturasra* (square), *dīrgha-caturasra* (rectangle) which are based on the relation of the sides and the terms *karṇa* (angle),

(e) Technical terms in the *Brāhmasphuṭasiddhānta* (ch. 12) of Brahmagupta (628 A. D.).

Terms	English Equivalents
1. *avalamba* (v. 22) or *lamba* (v. 23,25,33, etc)	Perpendicular (from the vertex to the base of a triangle.)
2. *anupāta* (v. 32)	Proportion.
3. *āyata* (v. 36)	Rectangle.
4. *bāhu* (v. 21)	Base or face.
(1) *pratibāhu* (v. 21)	Opposite side.
5. *bhūja* (v. 21,22,30 etc.)	Side.
6. *bhūmi* (v. 31)	Base.
7. *caturasra* (v. 23)	Square.
(i) *aviṣamacaturasra* (v. 23)	Square, rectangle or isosceles trapezium.
(ii) *āyatacaturasra* (v. 35)	Square of the side assumed at pleasure.
(iii) *dvisamacaturasra* (v. 36)	Quadrilateral having two equal sides.
(iv) *viṣamacaturasra* v. 29)	Quadrilateral having all unequal sides.
8. *caturbhūja* (v. 21)	Quadrilateral.
9. *ghana* (v. 6,7,18 etc.)	Volume.
10. *hṛdaya* (v. 27)	Semidiameter of a circle in contact with the angles of a triangle or quadrilateral.
11. *iṣu* (v. 42)	perpendicular.
12. *jātyā* (v. 38)	Right-angular.
13. *jīvā* (v. 41)	Chord.
14. *karṇa* (v. 23,24,25 etc.)	Diagonal.

eka-karṇa (a geometrical figure of equal angles), *dvi-karṇa* (a geometrical figure of two equal angles), *tri-karṇa* (triangle) etc. based on relations between the angles. In the *Sūryaprajñapati* (300 B. C.) both these types are represented respectively by *sama-caturasra* (square), *viṣama-caturasra* (quadrilateral other than square) having side basis, and *sama-catuṣkoṇa* (square or rectangle), *viṣama-catuṣkoṇa* (quadriangle other than square or rectangle) having angle basis. Both Āryabhaṭa I (c. 476 A.D.) and Brahmagupta (628 A. D.) however used the side basis depending on the terms *tribhūja*, *caturbhūja* and also *caturarasra* (square), *viṣama-caturasra* (quadrilateral having all unequal sides).

Euclid (c. 300 B. C.) in his *Elements* divides the rectilinear figures according to the number of their sides[1] but later on, he introduced the angle nomenclature also.[2] The Romans simply followed the Greek usage[3]. The early Egyptians together with the Babylonians, Hebrews and Arabs are said to have followed only the side nomenclature. Since the Indian School made use of both the angle and the side nomenclature from the time of the *Śulbasūtras* (c. 600 B. C.) i.e. earlier than Euclid, there is no reason to suggest that the Indians borrowed their ideas from those of the Greeks, on the other hand it might be otherwise.

Propositions, Areas and Miscellaneous Problems

(a) *Similar triangle* : In the *Śulbasūtras*, many similar figures were drawn, though there is no mention of

1. *Tri-pleuron, tetra-pleuron, poly-pleuron.*
2. *Tri-gonon, tetra-gonon.*
3. Tropfke, J. *Geschichte der Elementer Mathematik*, Bd. IV, p. 60–61, 1923.

the actual proportional relationship of the corresponding parts of the two similar figures. In the *Sūryasiddhānta* (c. 400 A. D.) and other astronomical treatises we meet with such proportional relationship between two similar figures.[1] The result like, "*Ksitijyā* (earth sine) multiplied by the *Vyāsārdha* (radius) and then divided by the *dina-vyāsadala* (day-radius) gives the *caradala* (ascensional difference)" is the direct application of this proportional relations of the similar triangles. Āryabhaṭa I (c. 476 A. D.) used this proportional relationship of the corresponding sides in case of similar triangles to solve shadow problems.[2] Similar application of many other astronomical and geometrical problems was found in the works of Brahmagupta[3] and other later scholars.[4]

(b) *The area of a triangle* : The theorems relating to the areas are found in Euclid (c. 300 B. C.) and apparently were common property long before his time. Egyptian surveyors, even before the time of Euclid were in the habit of finding the areas of the triangular and rectangular fields. Indians too, were able to calculate the area of a triangle ($= \frac{1}{2} \times$ base $\times$ altitude), rectangle (= length × breadth) and isosceles trapezium [$= \frac{1}{2}$ (sum of the parallel sides) × distance between the parallel sides] in connection with the construction of altars on the ground.[5] The same for-

1. Vide Burgess's edition of the *Sūryasiddhānta*, Calcutta, p. 61, 1935.
2. *Āryabhaṭīya* (*gaṇita*) v. 15 & 16.
3. *Brāhmasphuṭasiddhānta*, ch. 12, v. 54.
4. *Gaṇitasārasaṃgraha*, ch. 9; *Līlāvatī*, rule 238–45.
5. Vide, p. 121–22.

mulae are also found in the works of Āryabhaṭa I[1]. (c. 499 A. D.), and others[2]. For area of the scalene triangle[3] Bhāskara I (c. 600 A. D.) perhaps knew the area of a triangle with sides a, b and c to be $\sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a+b+c$. This formula may be obtained in terms of the sides as follows :

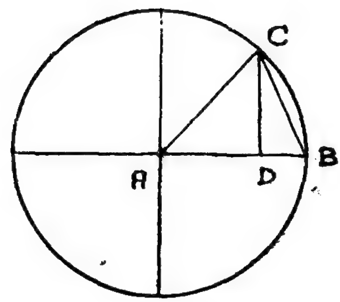


Fig. 12

$$AC^2 = AD^2 + CD^2$$
$$= (AB - BD)^2 + CD^2$$
$$= AB^2 + (BD^2 + CD^2) - 2AB.BD$$
$$= AB^2 + BC^2 - 2\ AB.BD$$

or $b^2 = c^2 + a^2 - 2.c.BD$

$$\therefore BD = \frac{c^2 + a^2 - b^2}{2c}$$
$$= \tfrac{1}{2}\left[c + \frac{a^2 - b^2}{c}\right] = x, \text{ say}$$

and

$$AD = c - x = \tfrac{1}{2}\left[c - \frac{a^2 - b^2}{c}\right]$$

According to Bhāskara I

$$CD = \sqrt{a^2 - x^2} \text{ or } \sqrt{b^2 - (c-x)^2}$$

$$\text{Now } CD = \sqrt{a^2 - x^2} = \sqrt{a^2 - \left(\frac{c^2 + a^2 - b^2}{2c}\right)^2}$$
$$= \sqrt{\frac{(2ac)^2 - (c^2 + a^2 - b^2)}{4c^2}}$$

1. *Āryabhaṭīya* (*gaṇita*), v. 6.
2. *Līlāvatī* of Bhāskara II, rule 164.
3. *Āryabhaṭīya bhaṣya*, ii, 6½; vide also Shukla, K. S. *Gaṇita*, 22, No. 2, p. 63, 1971.

$$= \sqrt{\frac{(2ac + c^2 + a^2 - b^2)(2ac - c^2 - a^2 + b^2)}{4c^2}}$$

$$= \sqrt{\frac{(c + a + b)(c + a - b)(b + c - a)(b - c + a)}{4c^2}}$$

$$= \sqrt{\frac{2s(2s - 2b)(2s - 2a)(2s - 2c)}{4c^2}}$$

$$= 4\sqrt{\frac{s(s - a)(s - b)(s - c)}{4c^2}}$$

Again,

$$\triangle ABC = \tfrac{1}{2}\ AB.CD$$

$$= \tfrac{1}{2}.\ c.\ 4\sqrt{\frac{s(s - a)(s - b)(s - c)}{4c^2}}$$

$$= \sqrt{s(s - a)(s - b)(s - c)}$$

This result perhaps was also known to Brahmagupta[1] and Mahāvīra.[2] The same formula is also found in the work of Bhāskara II[3] (c 1150 A. D.). A similar formula is given by the Greek author Heron[4] in the first century A. D.

(c) *The area of a cyclic* quadrilateral with sides a,b,c and d, is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $2s = a + b + c + d$. This formula is given by Brahmagupta[5]

1. *bhujakṛtyantarabhūrhṛtahīnayutābhurdvi bhājitā bādhe svābādhāvargonādbhujavargānmūlamvalamba* ।
(*Brāhmasphuṭasiddhānta*; ch. 12, v. 22).
2. *Gaṇitasārasaṃgraha*, ch. 7, v. 50.
3. *Līlāvatī*, rule 167.
4. Smith, D. E. *History of Mathematics*, 2, p. 287, Dover Publication.
5. *Brāhmasphuṭasiddhānta*, ch. 12, v. 21.

(c. 628 A. D.), Mahāvira[1] (c. 850 A. D.), Śrīdhara[2] (c. 850–950 A. D.) and Bhāskara II,[3] though these Indian scholars did not mention that the quadrilateral is cyclic. The proof may be obtained on the following lines.

From Fig. 12,

$\triangle ABC = \frac{1}{2}.\ AB.CD$

$= \frac{1}{2}.\ c.\ R\ Sin\ A = \frac{1}{2}\ c.b\ Sin\ A = \frac{1}{2}bc.\frac{R\ SinA}{R}$

[Vide chapter on Trigonometry.]

$= \frac{1}{2}.a.R\ \ Sin\ \ C = \frac{1}{2}ab\ Sin\ C$

$= \frac{1}{2}\ \frac{ab\ R\ SinC}{R}$ ....(1)

Again,

$BC^2 = CD^2 + BD^2$

$= CD^2 + (AB - AD)^2$

$= (CD^2 + AD^2) + AB^2 - 2AB.AD$

$= AC^2 + AB^2 - 2AB.AD$

or $a^2 = b^2 + c^2 - 2c.R\ \ Cos\ A$

$= b^2 + c^2 - 2c.b.\ \ Cos\ A$ ....(2)

$= b^2 + c^2 - 2bc.\ \frac{R\ Cos\ A}{R}.$

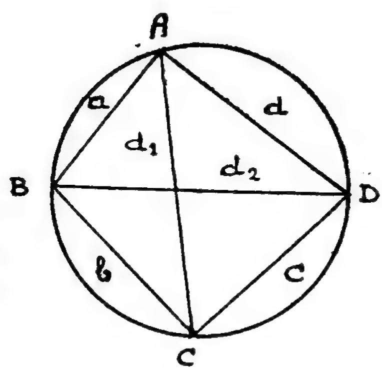


Fig. 13

Now, in Fig. 13, By (2) in $\triangle ABC$, AC^2

$= a^2 + b^2 - 2ab\ \frac{R\ Cos\ B}{R}$ ......(3)

1. *Gaṇitaśārasaṃgraha*, ch. 7, v. 50.
2. *Pāṭigaṇita*, sū., 117; (Ed. by Shukla).
3. *Līlavatī*, rule 167.

In $\triangle$ ADC,

$$AC^2 = c^2 + d^2 - 2cd \frac{R \text{ Cos } D}{R}$$

$$= c^2 + d^2 - 2cd \frac{R \text{ Cos}(180^\circ - B)}{R}$$

$$= c^2 + d^2 + 2cd \frac{R \text{ Cos } B}{R}$$

$$\therefore a^2 + b^2 - \frac{2ab \, R \text{ Cos } B}{R} = c^2 + d^2 + 2cd \frac{R \text{ Cos } B}{R}$$

$$\text{or } \frac{R \text{ Cos } B}{R} = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \quad \ldots \ldots \ldots \ldots (4)$$

$$\text{Now } (R \text{ Sin } B)^2 = R^2 - R^2 \text{Cos}^2 B = R^2 \left\{ 1 - \left(\frac{R \text{ Cos } B}{R} \right)^2 \right\}$$

$$= R^2 \left[1 - \left\{ \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \right\}^2 \right]$$

$$= R^2 \frac{\{2(ab + cd)\}^2 - (a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}$$

$$= R^2 \frac{\{(a + b)^2 - (c - d)^2\}\{(c + d)^2 - (a - b)^2\}}{4(ab + cd)^2}$$

$$= R^2 \frac{(a + b + c - d)(a + b - c + d)(c + d + a - b)(c + d - a + b)}{4(ab + cd)^2}$$

$$= R^2 \frac{2(s - d).\ 2(s - c).\ 2(s - b).\ 2(s - a)}{4(ab + cd)^2} \quad (\text{when}$$

$2s = a + b + c + d.$)

$$= R^2 \frac{16.(s - a)(s - b)(s - c)(s - d)}{4(ab + cd)^2}$$

$$\therefore \frac{R \text{ Sin } B}{R} = 2 \sqrt{\frac{(s - a)(s - b)(s - c)(s - d)}{(ab + cd)}} \quad \ldots \ldots (5)$$

Now quad. ABCD

$= \triangle ABC + \triangle ADC$

$$= \frac{1}{2}\, ab \frac{R \text{ Sin } B}{R} + \frac{1}{2}\, cd \frac{R \text{ Sin } D}{R} \quad [\text{ From } (1)\ ]$$

$$= \frac{1}{2}\, ab \frac{R \text{ Sin } B}{R} + \frac{1}{2}\, cd \frac{R \text{ Sin } (180^{0} - B)}{R}$$

$$= \frac{1}{2}\, ab \frac{R \text{ Sin } B}{R} + \frac{1}{2}\, cd \frac{R \text{ Sin } B}{R}$$

$$= \frac{1}{2}(ab + cd)\frac{R \text{ Sin } B}{R}$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad [\text{ From } (5)\ ]$$

Āryabhaṭa II (c. 950 A. D.) has however pointed out that this rule does not give the accurate area of any quadrilateral. (*Mahāsiddhānta*, xv. 69).

(d) *Diagonals of a quadrilateral* are given by Brahmagupta[1] (628 A. D.), Mahāvīra[2] (c. 850 A. D.) and Bhāskara II[3] (c. 1150 A. D.). They gave the following lengths of the diagonals of a quadrilateral in terms of the given sides, which is valid only for a cyclic quadrilateral. If a, b, c, d be the sides in order, and d_1 and d_2 be the two diagonals, then :

$$d_1 = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \text{ and}$$

$$d_2 = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$$

The proof is simple and follows from result (2) under Fig. 13.

We have, $AC^2 = a^2 + b^2 - 2ab\frac{R \text{ Cos } B}{R}$

1. *Brāhmasphuṭasiddhānta*, ch. 12, v. 28.
2. *Gaṇitasārasaṃgraha*, ch. 7, v. 54.
3. *Līlāvatī*, rule 190.

$$= a^2 + b^2 - 2ab.\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \text{[From 4]}$$

$$= \frac{2(ab + cd)(a^2 + b^2) - 2ab(a^2 + b^2 - c^2 - d^2)}{2(ab + cd)}$$

$$= \frac{2cd(a^2 + b^2) + 2ab\,(c^2 + d^2)}{2(ab + cd)} = \frac{(ac + bd)(ad + bc)}{ab + cd}$$

$$\therefore AC = d_1 = \sqrt{\frac{(ac + bd)\,(ad + bc)}{ab + cd}}$$

Similarly it may be shown,

$$BD = d_2 = \sqrt{\frac{(ac + bd)(ab + cd)}{ad + bc}}$$

Again, $d_1^2.d_2^2 = (ac + bd)^2$

$\therefore d_1.d_2 = ac + bd.$

The formulae were obtained by Ptolemy (c. 150 A. D.), though the exact method followed by him is not known. The first of these formulae i.e. the expression for d_1 was rediscovered by W. Snell[1] in 1619 A. D. Brahmagupta's method for the determination of the diagonals of a cyclic quadrilateral constitutes a remarkable contribution in geometry of the time.

(e) *The circum radius of a triangle* as given by Brahmagupta[2] is $\frac{bc}{2p}$ wheıe b and c are two sides of the triangle and p is the triangle drawn through their point of intersection. No proof is given. The proof is obvious In any triangle, AD = R Sin B (vide Fig. 12).

$= c \text{ Sin } B$

$$= \frac{c.R \text{ Sin } B}{R}$$

1. Smith, D. E. *History of Mathematics*, 2, p. 287, Dover Publication.
2. *Brāhmasphuṭasiddhānta*, ch. 12, v. 27.

$$= \frac{c \,.\, b}{R \,.\, 2}\left[\because \frac{a}{R \text{ Sin } A} = \frac{b}{R \text{ Sin } B} = \frac{c}{R \text{ Sin } C} = 2 \text{ in a circumscribed circle,} \right]$$

$$\therefore \; p = AD = \frac{bc}{2R}$$

$$\text{or } R = \frac{bc}{2p}$$

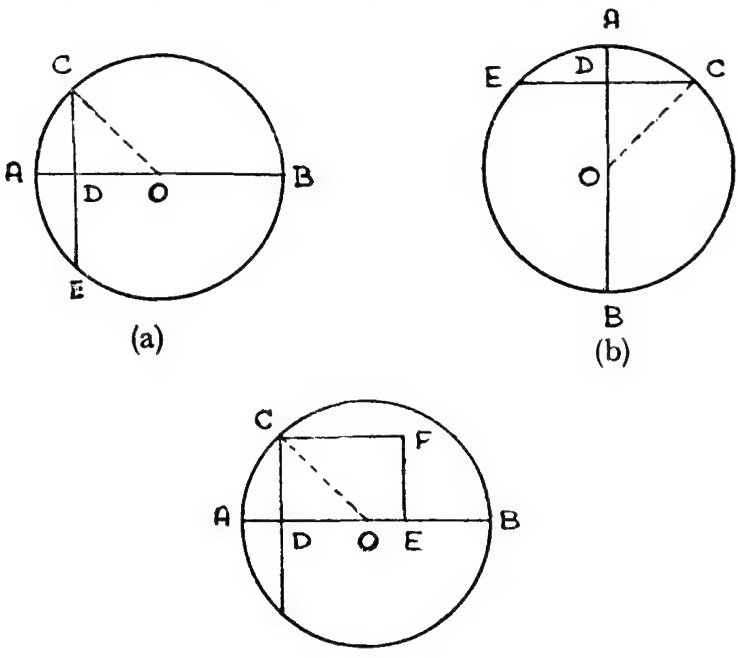
(f) *Area of a circle* is given by Āryabhaṭa[1] I and and Bhāskaia II[2] as $\frac{1}{2}$ c.r. = πr^2 where c = circumference and r = radius. Gaṇeśa gives a very simple proof in which he has divided a circle into even number of equal triangles by drawing radius to the circumference. Half of the total number of triangles are so fitted with remaining triangles that they form a rectangular area having breadth = radius, and length = half the circumference i.e. area = $r.\frac{1}{2}c = \pi r^2$.

(g) *The surface area*[3] of a sphere was given Bhāskara II as $4\pi r^2$. According to him the area of a circle multiplied by four is the net covering the surface of the ball having the same radius.

(h) *The value of the chord of a circle* is given by Āryabhaṭa I[4] by the relation as $CD^2 = AD.DB$, when the chord CDE intersects the diameter ADB (Fig. 14a.) of a circle. CD has been defined as R Sin BC or R Sin AC by Bhāskara I (vide his *Āryabhaṭīyabhāṣya*). This rule is given by Brahmagupta[5] (c. 628 A.D.) and

1. *Āryabhaṭīya* (*Gaṇita*), v. 7.
2. *Līlāvatī*, rule 203.
3. *Līlāvatī*, rule 203.
4. *Āryabhaṭīya* (*Gaṇita*), v. 17½.
5. *Brāhmasphuṭasiddhānta*, ch. 12, v. 41.

Bhāskara II.[1] In justification of this rule Bhāskara I has selected several interesting problems such as the hawk and rat problems, the lotus problems and the



(c)
Fig. 14

crane and fish problems etc. Some of these problems also occur in later works. The problems are as follows :

Problem 1 : "A hawk is sitting on the top of a pole whose height is 18 (cubits). A rat who has gone out of his dwelling (at the foot of the pole) to a distance of 81 (cubits), while returning towards his dwelling,

1. *Līlāvatī*, rule 206-7.

afraid of the hawk, is killed by the cruel (bird) on the way. Say how far has he gone towards his hole, and also the (horizontal) motion of the hawk (the speeds of the rat and the hawk being the same". (*Āryabhaṭīya Bhāṣya*, ii, 17; reappeared also in Yallaya's commentary).

Bhāskara I ascribes such problems to previous writers. His method of solution runs as follows :

Let AOB be the horizontal diameter of the circle of centre O. CDE, the vertical chord intersects AOB at D. Imagining that the hawk is sitting at C and the rat at B, they see each other. The rat then runs to his hole at D, but is killed by the hawk at O. Here CO = BO, Since speed of the hawk and rat are same. Given CD = 18 cubits, DB = 81 cubits, since $CD^2 = AD \times DB$, AD = 4 cubits.

$$\begin{aligned} \therefore\ DO &= DB - OB \\ &= DB - OA \\ &= DB - (DO + AD) \\ \text{or}\quad DO &= \tfrac{1}{2}(DB - AD) \\ &= \tfrac{1}{2}(81 - 4) = \tfrac{1}{2}\,.\,77 \\ &= 38\tfrac{1}{2} \text{ cubits} \end{aligned}$$

$$\text{Similarly } BO = \tfrac{1}{2}(DB + AD) = \tfrac{1}{2}(81 + 4) = \tfrac{1}{2}.85 = 42\tfrac{1}{2} \text{ cubits}$$

Hence the rat has crossed a distance of $42\frac{1}{2}$ cubits, and the horizontal movement of the hawk along CO is DO = $38\frac{1}{2}$ cubits.

Problem 2 : "A lotus flower of 6 *aṅgulas* just dips (into the water) when it advances through a distance of 2 cubits. I want to know the height of the lotus plant and the depth of the water." (Similar problems occur in the works of Bhāskara II and Narāyaṇa).

In this problem (Fig. 14b), AOB is the vertical diameter of the circle of centre O and EDC, the horizontal chord, denotes the water level. OA is the lotus stalk, AD lotus flower, and O the root of the lotus, and E and C the points where the lotus flowers just dips into the water. Here AD = 6 aṅgulas = $\frac{1}{4}$ cubits DC = 2 cubits and $DB = \frac{DC^2}{AD}$

$$= \frac{2^2}{\frac{1}{4}} = 2^2 \times 4 = 16 \text{ cubits.}$$

Like that of previous problem,

$$OA = \text{height of the plant} = \tfrac{1}{2}\ (DB + AD)$$
$$= \tfrac{1}{2}\ (16 + \tfrac{1}{4})$$
$$= \frac{65}{8} = 8\tfrac{1}{8} \text{ cubits}$$

$$OD = \text{depth of the water}$$
$$= OA - AD = \tfrac{65}{8} - \tfrac{1}{4} = 7\tfrac{7}{8} \text{ cubits.}$$

Problem 3 : "There is a reservoir of water of dimensions 6 × 12. At the east-north corner there is a fish and at the west-north corner there is a crane. For fear of him, the fish, crossing the reservoir, hurriedly went towards the south in an oblique direction but was killed by the crane who came along the sides of the reservoir. Give out the distances travelled by them assuming that their speeds are the same." (*Āryabhātiya bhaṣya*, ii. 17; reappears in the Raghunatha Raja's commentary. See also *Gaṇitakumudī*, Kṣetravyāhara, p. 38–39, Ex. 29–31).

Let CDEF be the reservoir (Fig. 14c) of which DE = CF = 12 and CD = FE = 6, suppose C = east-north corner, and F = west-north corner. At first the fish was at C and the crane at F. Then out of fear the fish swims along CO and the crane moving along FE and EO, kills the fish at O. Then CO = FE + EO = EB + EO (EB is drawn equal to FE).

A circle is drawn with O as centre and OC as radius, which will pass through B.

We have $DB = DE + EB$
$= DE + FE$
$= 12 + 6 = 18$

Let DB produced meet the circle at A.

$\therefore AD = \frac{CD^2}{DB} = \frac{6^2}{18} = \frac{36}{18} = 2$

$\therefore AB = AD + DB = 2 + 18 = 20$

$\therefore OC = 10$

Therefore distance covered by the crane = 10 = distance covered by the fish.

(i) *The volume of solid bodies.* Āryabhaṭa I's formula for the volume of a sphere and that of a triangular pyramid are not quite correct.[1] His formula for a sphere is $\sqrt{\pi}.\pi r^3$ or $1.47\pi r^3$, the correct value being $\frac{4}{3}\pi r^3$ or $1.33.\,.\pi r^3$. The correct value was given by Bhāskara II.[2] The formula for that of the triangular pyramid is given incorrectly by Āryabhaṭa I (*Gaṇita-pāda,* verse 8) as $\frac{1}{2} \times$ area of the triangular base $\times$ height. The correct formula for a square pyramid is given by Bhāskara II as $\frac{1}{3} \times$ radius $\times$ square base. For volume of the sphere, Bhāskar II has divided it into as many pyramids as there are unit square areas on the surface, its height being equal to the radius. Hence

1. *Āryabhaṭīya* (*Gaṇita*), v. 7.
2. *Līlāvatī*, rule 203.

the volume of the sphere becomes equal to $\frac{1}{3}$ × radius × surface of the sphere i.e. $\frac{4}{3}\pi r^3$.

The expression for the volume of a cone occurs first in India in the work of Brahmagupta. He used the term *sūci*, meaning thereby generally a pyramid with a base of any form. The base may be a circle and hence the term includes a cone as well. According to Brahmagupta[1] (c. 628 A. D.) *volume of a cone* (*or pyramid* = $\frac{1}{3}$ × area of the base × height. This formula reappears in the works of Mahāvīra[2] (c. 850 A. D.), Āryabhaṭa II[3] (c. 950 A. D.), Nemichandra[4] (c. 975 A. D.), Śrīpati[5] (1039 A. D.) and Bhāskara II[6] (1150 A. D.). The formula refered to by Kaye[7] is $\left(\frac{\text{Circumference}}{6}\right)^2$ × height. This formula no doubt occurs in the works of the above writers except Mahāvīra. But in connection with this formula it has beeu specifically mentioned by all of them that the formula is to be employed only in case of "the measurements of the maunds grain" (*rāśi-vyāvahāra*), not in other cases. It has been further remarked by Brahmagupta, Śrīpati and Bhāskara II that in that case the height of the maund must be assumed to be equal to the circumference of the base divided by 9, 10 or 11 according to

1. *Brāhmasphuṭasiddhānta*, ch. 12, v. 44.
2. Compare *Gaṇitasārasaṃgraha*, ch. 8, v. 17½ and 20½.
3. *Mahāsiddhānta*, ch. 15, v. 105.
4. *Trilokasāra* of Nemichandra, Gāthā 19.
5. *Siddhāntaśekhara*, ch. 13, v. 44.
6. *Līlāvatī*, vide Sudhākara Dvivedī's edition, p. 65; Colebrooke, *Hindu Algebra*, p. 98.
7. Kaye, G. R. *Indian Mathematics*, p. 39.

calculates two approximate values called *vyavahārika gaṇita* or 'approximate value' (A) and *antra gaṇita* or 'gross volume' (G). Mahāvīra calls them respectively *karmāntika-phala* and *auṇḍra-phala.* It is stated : A = (area from half the sum of the linear dimensions of face and base) × height, and G = ½ (area of face + area of base) × height. The accurate volume (V) of the frustum is then given to be $V = \frac{1}{3}(G - A) + A$. Now for the frustum of a right circular cone noted before,

$$A = \frac{(r+R)^2}{2}\pi \times h, \quad G = \frac{1}{2}(\pi r^2 + \pi R^2) \times h$$

$$\therefore V = \frac{\pi}{3}(r^2 + R^2 + rR) \times h \text{ as stated before.}$$

For a *volume of a frustum of a pyramid* the sides of whose upper face are a, b and the corresponding sides of whose lower face are a′, b′, the approximate value will be

$$A = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right) \times h, \quad G = \frac{1}{2}(ab + a'b')h$$

Then the accurate volume of the frustum of the wedge will be

$$V = \frac{1}{3}\left[\frac{1}{2}(ab + a'b')h - \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h\right] + \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

On reduction we easily obtain,

$$V = \frac{1}{6}\left[ab + a'b' + (a+a')(b+b')\right] \times h \ldots \ldots (2)$$

The formula reappears in this reduced form in the works of Āryabhaṭa II[1], Śrīpati[2] and Bhāskara.II[3]

1. *Mahāsiddhānta*, ch. 15, v. 106.
2. *Siddhāntaśekhara*, ch. 13, v. 45.
3. *Līlāvatī*, Ibid., p. 45.

the kinds of grain. Nemicandra has considered the case (for finer grains) in which the height is one-eleventh of the circumference of the base.[1] So there is absolutely no doubt that the formula was intended only for a rough calculation.

The formula of the *volume of a frustum* appears explicitly in India first in the works of Śrīdhara.[2] The frustum is the part of a solid figure such as a cone or a pyramid, cut off by a plane parallel to the base. If d and D denote the diameters of the upper and lower face of the frustum of a right circular cone and h its height, then its volume V will be given by,

$$V = \frac{h}{24}\sqrt{10}\,[\,d^2 + D^2 + (d + D)^2\,]$$

$$= \frac{\pi}{3}\,(r^2 + R^2 + rR)h$$

where r, R denote the radii of the upper and lower face and $\pi = \sqrt{10}$, the value adopted by Śrīdhara. It also reappears in the works of Āryabhaṭa II[3] and Mahāvīra.[4] Brahmagupta[5] followed by Mahāvīra[6] gave a general method of calculating the volume of frustum of a solid, such as a pyramid, a cone and a wedge whose upper and lower faces are parallel. He first

1. Gāthā 23, in which Nemicandra has specified the kinds of seeds for which this assumption is to be made.
2. *Triśatikā*, rule 58. This clearly shows that Śrīdhara knew how to find the volume of a complete right circular cone though he had not explicitly recorded it.
3. *Mahāsiddhānta*, ch. 15, v. 106.
4. *Gaṇitasārasaṃgraha*, ch. 8, v. 17½.
5. *Brāhmasphuṭasiddhānta*, ch. 12, v. 45-6.
6. *Gaṇitasārasaṃgraha*, ch. 8, v. 9-11½.

The formula for the volume of a truncated wedge (frustum of a pyramid) is given in the *Chiu-Chang Suan-Shu*[1] and *Mong-hsia Pi-t'an*[2] of Ch'en Huo (died in 1075 A. D.) as

$$\tfrac{1}{6}\,[\,(2a+a')\,b+(2a'+a)\,b'\,]\times h \quad \dots \quad \dots \quad \dots (3)$$

This expression is of course easily reducible to the second Hindu form. But the first Hindu form i. e. the Brahmagupta-Mahāvīra form is considerably different from the Chinese form. Putting $a=b$, $a'=b'$ in (3), we get the formula for the volume of a frustum of a pyramid with a square base as $\frac{1}{3}(a^2+a'^2+aa')\,h$. This particular formula occurs in the *Chiu Chang Suan-Shu*[3], *Chang Ch'iu-Chien Suan-Ching*[4] and Heron's *Stereometry*.[5] It was also known to the ancient Egyptians.[6] Cantor[7] followed by Tropfke[8] and Smith[9] thinks that Brahmagupta's rule was meant for such a particular case only. But we do not think so.[10] For there is nothing in Brahmagupta's definition of his rule to suggest such a limitation. These writers were

1. Mikami, Y. *The Development of Mathematics in China and Japan*, p. 15, 1913.
2. *Ibid.*, p. 61.
3. *Ibid.*, p. 15.
4. *Ibid.*, p. 42.
5. Heath, T. L. *History of Greek Mathematics*, 2, p. 334.
6. Touraeff, B. *Ancient Egypt*, p. 100, 1917.
7. Cantor, M. *Vorlesungen über Geschichte der Mathematik*, Bd. 1, p. 649, Leipzig, 1907.
8. Tropfke, J. *Geschichte der Elementer Mathematik*, Bd. 7, p. 24, fn. Leipzig, 1924.
9. Smith, D. E. *History of Mathematics*, 2, p. 293.
10. Sudhākara Dvivedī also is of the same opinion. *Vide* his notes on Brahmagupta's rule.

probably misled by an example of a well with a square face (a case of a truncated square pyramid turned upside down) given by the commentator Pṛthudakasvāmī in illustration of Brahmagupta's rule. Mahāvīra's rule is absolutely free from any kind of limitation. For an illustration of it, he has given examples relating to a truncated pyramid with a triangular, square and rectangular base, a frustum of a right circular cone and also a truncated wedge.[1]

Varieties of Plane Jigures

Mahāvīra enlists three varieties of triangles (equilateral, isosceles, and scalene), five varieties of quadrilaterals (equilateral, equidichastic, equibilateral, equitrilateral and inequilateral) and eight varieties of curvilinear figures (circle, semi-circle, ellipse, conchiform area, concave circle. convex circle, out - lying annulus, and in-lying annulus), while Śrīdhara takes the primary plane figures to be ten in number. They are rectangular (*āyata*), equilateral (*sama bhuja*), equibilateral (*dvi-sama bhuja*), equi-trilateral (*tri-sama bhuja*), the inequilateral quadrilateral (*visamabhuja-caturaśra*), the equilateral triangle, the scalene triangle, the isosceles triangle, the circle (*vṛtta*) and the segment of a circle (*vṛtta cāpa*).[2] Nārāyaṇa's list of plane figures are also ten in numbers though he has replaced 'the segment of a circle' from the list of Śrīdhara by the "conch-figure". According to Śrīdhara the area of these figures are

1. *Gaṇitasārasaṃgraha*, ch. 8, v. 12½-18½. There is obviously an error in the example relating to the truncated wedge (ch. 8, v. 16½). 32 should be 22.
2. *Pāṭīgaṇita*, sū. 110–11 (Ed. by K. S. Shukla)

determined by their own rules, and the areas of other figures are determined by considering their shapes in terms of the plane figures.

Some Geometrical Constructions from Pythagoras Theorem

(a) *A right-angled triangle* can be easily constructed from Pythagoras' theorem, when the value of any two sides are known.

This has been expressed by algebraical formula by Brahmagupta, Mahāvīra and Bhāskara II.[1] The deduction is obvious from Pythagoras' Theorem.

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1. Brahmagupta gives the general solution (ch. 12, v. 33) for the rightangled triangle $a^2+b^2=c^2$ as $a=2pq$, $b=p^2-q^2$, $c=p^2+q^2$. His other solution (ch. 12, v. 35) for the given side a is :

$a, \frac{1}{2}\left(\frac{a^2}{p}-p\right), \frac{1}{2}\left(\frac{a^2}{p}+p\right)$ where p is any rational number.

Mahāvīra's solution: $a, \frac{2qa}{q^2-1}, \left(\frac{q^2+1}{q^2-1}\right)a$. Putting $p=\left(\frac{q-1}{q+1}\right)a$ in Brahmagupta's second result we get Mahāvīra's rightangled triangle. The same solution is also given by Bhāskara II (*Līlāvatī*, rule 139). Mahāvīra (*Gaṇitasārasaṃgraha*, ch. 7, v. $222\frac{1}{2}$) constructs another rightangled triangle for the known hypotenuse c as : $c, \frac{2pqc}{p^2+q^2}$ $\left(\frac{p^2-q^2}{p^2+q^2}\right)c$, Bhāskara's solution (*Līlāvatī*, v. 142) is : $c, \frac{2nc}{n^2+1}, \left(\frac{n^2-1}{n^2+1}\right)c$. This readily follows from Mahāvīra's solution by putting $\frac{p}{q}=n$. Dickson is evidently unaware of the solution given by Brahmagupta and Mahāvīra, for he ascribes the Brahmagupta and Mahāvīra results to Leonardo Fibonacci (c. 1202 A.D.) and Vieta (1580 A.D.) Dickson — *History of Theory of Numbers*, **2**, p. 192–3).

(b) *Construction of scalene triangle* – In the Vedic period we have clearly pointed out, how a brick having the shape of a scalene triangle can be constructed by fusing two right angular bricks joining any two equal sides. The same principle has been adopted by Brahmagupta, Mahāvīra and Bhāskara II by the application of Pythagoras' theorem in algebraical form.[1]

(c) *Construction of a quadrilateral* – A quadrilateral can be constructed, i) by joining two scalene triangles of equal bases along their base, ii) by joining four such

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1. Brahmagupta (*Brāhmasphuṭasiddhānta*, ch. 12, v. 34) divides the scalene triangle ABC (Fig. 15) by drawing a perpendicular AD from the vertex A. The perpendicular AD evidently divides the triangle ABC into two right-triangles ADB and ADC. If AD= p, then from the previous knowledge, AB $= \frac{1}{2}\left(\frac{p^2}{m} + m\right)$, BD$=\frac{1}{2}\left(\frac{p^2}{m} - m\right)$ and AC$=\frac{1}{2}\left(\frac{p^2}{n} + n\right)$ and DC$=\frac{1}{2}\left(\frac{p^2}{n} - n\right)$ where m, n are national numbers. Hence BD $= \frac{1}{2}\left(\frac{p^2}{m} + \frac{p^2}{n} - m - n\right)$. Mahāvīra (850 A.D.) says (Datta, B. and Singh, A.N. *Ibid.*, 2, p. 225–227) that if $m^2 - n^2$, 2mn, $m^2 + n^2$ and $p^2 - q^2$, 2pq, $p^2 + q^2$ be the sides of the two rightangled triangle and AD is so chosen that AD = 2mn=2pq, then the two triangles may be taken as ABD, ACD or as AED, ACD where ABD=AED. Then the triangle is ABC or AEC. The problem and its solutions are given in Europe by Buchet (1621 A.D.) and Cunliffe (Dickson, *History of Theory of Numbers*, 2, p. 192–3).

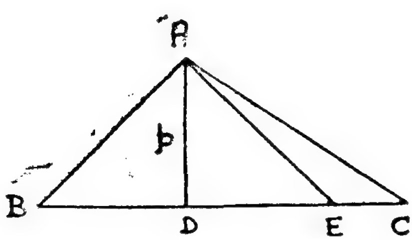

Fig. 15

right-triangles in a way that each pair forms a scalene triangle with equal bases. Such pairs can be constructed by suitable selection of right-angled triangle whenever necessary and iii) by joining four scalene triangles. The number (iii) is not dealt by the Hindu mathematicians, but the algebraical formulae for determining the values of the quadrilateral of number (i) and (ii) were reduced from the Pythagoras' theorem by Brahmagupta.[1] Brahmagupta's method of constructing a quadri-

1. Let a_1, b_1, c_1 and a_2, b_2, c_2 be the sides of the two right-angled triangles such that $a_1^2+b_1^2=c_1^2$ and $a_2^2+b_2^2=c_2^2$.

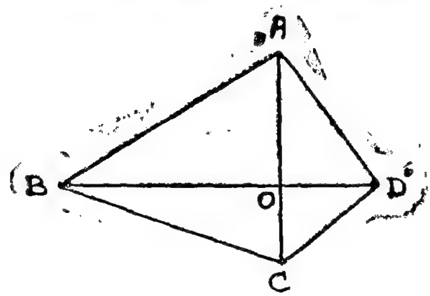

Fig. 16

Now two triangles viz. COD (Fig. 16) with sides $a_1(a_2, b_2, c_2)$ and AOD with sides $a_2(a_1, b_1, c_1)$ are made and joined with their common side OD $(=a_1 a_2)$. On the other side of AC two triangles viz. AOB with sides $b_1 (a_2, b_2, c_2)$ and BOC with sides $b_2 (a_1, b_1, c_1)$ are made and joined with their common side $OB = b_1 b_2$. This gives a quadrilateral ABCD with sides $b_1 c_2$, $b_2 c_1$, $a_1 c_2$, $a_2 c_1$, in order. The diagonals of a quadrilateral are at right angles. From (e), p. 53, the circumradius of the triangle ABC $= \frac{b_2 c_1 \times b_1 c_2}{2b_2 b_1} = \frac{c_1 c_2}{2}$.

Again circumradius of the triangle ABD $= \frac{b_1 c_2 \times a_2 c_1}{2a_2 b_1} = \frac{c_1 c_2}{2}$

This shows that the quadrilateral ABCD is cyclic and the radius is $\frac{c_1 c_2}{2}$. The result is obvious from Brahmagupta's conception of forming a scalene triangle by joining two right-triangles. Brahmagupta gives another set of solution, i.e. $c_1 (a_2, b_2)$, $c_2 (a_1, b_1)$ or $c_1 a_2$, $c_1 b_2$, $c_2 a_1$, $c_2 b_1$ are lateral was indeed a notable achievement in this field. An instance of constructing a quadrilateral by Brahmagupta's method is given by Pṛthudakasvāmīn[1] (c 850 A. D.) and Śrīdhara[2] (c. 900 A. D.). Bhāskara II[3] (1150 A. D.) reproduced the method of Brahmagupta.

Similar application of Pythagoras' theorem has been made by the same authors for i) the construction of a quadrilateral whose opposite sides are equal[4] and ii) the construction of a trapezium when three sides are equal.[5]

Some Typical Constructions

Some problems are given by Mahāvīra for the construction of a rectangle which require the knowledge of the solution of linear, quadratic, indeterminate equations of 1st and 2nd degree. Mahāvīra gave only the results. Datta[6] gave a complete solu-

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the opposite sides of another convex quadrilateral (Vide *Brāhmasphuṭasiddhānta*, ch. 12, v. 38). Brahmagupta previously mentioned the term *caturbhūjakoṇaspṛsagvṛtta* which means that he probably considered "an inscribed quadrilateral whose vertices touch the circle" (*Ibid.*, ch. 12, v. 27).

1. Two right-triangles viz. (3, 4, 5) and (5, 12, 13) were taken to construct the quadrilateral (Vide Colebrooke, H. T.-*Algebra with Arithmatic and Mensuration from the Sanskrit of Brahmagupta and Bhascara* p. 307, fn. 7, London, 1817).
2. *Triśatikā*, Ex. 80.
3. *Līlāvatī*, rule 191–92.
4. *Brāhmasphuṭasiddhānta*, ch. 12, v. 36; *Gaṇitasārasaṃgraha* of Mahāvīra, ch. 7, v. 99½.
5. *Brāhmasphuṭasiddhānta*, ch. 12, v. 37; *Gaṇitasārasaṃgraha*, ch. 8, v. 101½.
6. Datta, B. 'On Mahāvīra's solutions of rational triangles and quadrilaterals', *Bulletin of the Calcutta Mathematical*

tion depending on the results of Brahmagupta, Mahāvīra, Bhāskara II and Nārāyaṇa. Some of the problems run as follows :

(a) Construction of a rectangle whose area is numerically a multiple of the perimeter or the diagonal, or in general, a linear combination of the sides and the diagonal.

(b) Construction of a rectangle whose perimeter is one (1).

(c) Construction of two rectangles, whose perimeters are equal, but the area of one is double that of the other.

(d) Construction of two rectangles, whose areas are equal but the perimeter of one is double that of the other.

(e) Construction of two rectangles, the perimeter of one is double that of the other and the area of the latter is double that of the former.

Foundation of Solid (Co-ordinate) Geometry

Brajendranath seal[1] first of all attracted our attention to the fact that the ancient Indian scholar Vācaspati[2] (841 A. D.) led the foundation of solid co-

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Society, **20**, p, 287, 1930; Datta, B. and Singh, A. N. *History of Hindu Mathematics*, **2**, p. 228–248, Motilal Banarsi Das, Lahore, 1938; Srinivasienger, C. N. *The History of Ancient Indian Mathematics*, p. 75–78, Calcutta, 1967.

1. Seal. B. *The Posiitve Sciences of the Ancient Hindus*, Motilal Banarasi Das, p. 117, 1958.
2. For date vide *Gleanings from the History and Bibliography of the Nyāya Vaiśeṣika Literature by* Gopinath Kaviraj, p. 14, Calcutta, 1961.

ordinate geometry. The conception developed from the study of the position of the atom in space. This has been stated in his *Tātparyaṭīkā*[1], a commentary by Vācaspati on Uddyotakara's *Nyāyavārtika*. To conceive position in space Vācaspati takes three axes, one proceeding from the point of sunrise in the horizon to that of the sunset, on any particular day (roughly speaking from east to west). The second bisects this line at right angle on the horizontal plane (roughly, north-south line). The third is drawn from the point of their interesection upto the meridian position of the sun on that day (roughly up-down line). The position of any point in space relatively to another point is measured by its distances alongthese three axes i. e. by arranging in a numerical series the intervening points of contact, the less distance will come earlier and the greater distance will come later in the series. This gives only a geometrical analysis of the conception of the three-dimensional space. But it must be admitted that in a rudimentary manner it anticipates the foundation of solid (co-ordinate) geometry.

1. *ektve'pī diśaḥ ādityodayadeśapratyāsannadesasaṃyukto yaḥ sa itarasmādviprakṛṣṭadeśasaṃyogātparamāṇoḥpūrvaḥ* । *evam-ādityāstamayadeśa paramāṇu apekṣya yaḥ sūryodayāstamaya-deśa viprakṛṣṭadeśasaṃyogaḥ sa madhyavartī* । *evamentayor yau tiryagdeśa saṃbandhinau madhyasya arjavena vyavasthitau pūrśvavartinau tau dakṣiṇottarau paramāṇu evaṃ madhyadina-vartisūryasannikarṣa viprakarṣāvupekṣya uparyadhobhāvo draṣṭavyaḥ* । *saṃyuktasaṃyogālpatvabhuyastve ca pratyāsanna-deśasamyukto yaḥ sa itarasmādviprakṛṣṭadeśa samyogātpar amāṇoḥ paścimaḥ, tau ca pūrvapaścimau sannikarṣa viprakar-ṣau, pūrvasaṃkhyāvacchinnatvaṃ parasaṃkhyāvacchinnatvaṃ ca bhūyastvam* । (*Tātparyaṭīkā*, ch. 4, Ahnik 2, Sūtra 25)

Euclid's Elements in India

Prince Maḥmud Shāh Bahamani (2nd half of 14th century A. D.) studied Taḥrīr - au - Uqlīdas, Ar-bian version of Euclid's *Elements*.[1] Abul Fazl (c. 1590 A. D.) in his *Ain-i-Akbari* has referred to some of the propositions of the Euclids' *Elements*. He has compiled it in a way which shows that he had thorough acquaintance with the work.[2] Abul Fazl got his education in India, so he must have learnt Euclid's *Elements* here, of course, in Arabic or in Persian translation. This hints that the work was known in India at least in the 14th century A. D. We do not know exactly, when Euclid's *Elements* came to India but we are sure that in the beginning it remained confined to the circle of Moslem scholars in India.

The earliest trace of its influence is found in the Sanskrit work, *Siddhāntatattvaviveka* ot Kamalākara, the court astronomer of the emperor Jahangir of Delhi. Some of the passages of *Siddhāntatattvaviveka* are evidently copied from the *Elements*.[3] Other instances of resemblance

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1. *Ferishta*, translated by John Briggs, 2, p. 296.
2. 2,p. 415–6; 3, p. 24 (Jarrett's edition).
3. The definition and classification of a line is given in the following terms, which is very akin to that of Elements of *Euclid* ;

dairghaṃ yasyāḥ sadaivāsti vistāro naiva labhyate ।
atisūkṣmā ca sā rekhā jñeyā budhimatā dvidhā ॥
avakrā vakragā tatra'vakrā tu saralābhidhā. ।

(*Siddhāntatattvaviveka, ch*. 3, v. 22–23½)

"Of which there is always the length, the breadth does not can also be identified with particular propositions of *Elements* without much difficulty.[1] On the whole, there is absolutely no doubt that Kamalākara had knowledge of Euclid's *Elements*. The first complete translation of this work in Sanskrit was made in 1718 A. D. under the title *Rekhāgaṇita* ("Calculation with lines") by Samrāṭ Jagannātha at the command of his patron king Jayasiṃha of Jaipur. It was rendered from the Arabic translation of Euclid by Naṣir-ed-dīn aṭ-ṭusi (c. 1250 A. D.). The *Rekhāgaṇita* contains 15 *adhyāyas* or Books. Books I-IV are devoted to plane geometry and Books XI–XIV treat of solid geometry. The remaining books deal with laws of proportion, principle of numbers and other related problems,

In the *Rekhāgaṇita*, we find quite a large number of Sanskrit words coined by the author for translating Euclid's *Elements* from its Arabic version. A list of these terms is tabulated below with their English equivalents :

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exist and (which is) very fine. That line should be known by the intelligent as of two kinds : curved and uncurved. Of these the uncurved is called straight......".

1. For instance, compare proposition of III. 38, 45, 46, 48 with those of Euclid's *Elements* – I. 21; VI. 8; I. 15; I. 19 respectively.

(a) Technical terms of *Rekhāgaṇita* connected to plane figures :

Sanskrit terms	English Equivalents
1. *abhīṣṭa-rekhā*	Straight line of desired length.
2. *āyata*	Oblong.
3. *cāpa–karṇa*	Chord.
4. *kodaṇḍa*	Segment of a circle.
5. *koṇa*	Angle.
i *adhikakoṇa*	Obtuse angle.
ii) *alpakoṇa*	Acute angle.
iii) *samakoṇa*	Right angle.
iv) *viṣama-koṇa*	Angle other than a right angle.
6. *kṣetra*	Figure.
i) *dharātalakṣetra*	Superficies.
ii) *sama-dharātalakṣetra*	Plain superficies.
iii) *viṣama-dharātalakṣetra*	Crooked superficies.
	N. B. – From one point to another number of crooked lines may be drawn, but one right line which is the shortest may be drawn. So infinite number of crooked superficies may be drawn from one line to another, but only one plain superficies may be drawn which is shortest.
7. *nyāsa*	Construction.

Sanskrit Terms	English Equivalents.
8. *nyunakoṇa*	An acute angle.
9. *pāli*	Circumference.
10. *rekhā*	Line
i) *samānāntara–rekhā*	Parallel lines. The definition omits the most important thing that parallel lines must be in the same plane.
ii) *sarala rekhā*	Right line or straight line.
iii) *viṣama-rekhā*	Crooked line.
11. *vṛtta-kṣetra*	Circle.

(b) Technical terms for solid figures :

Sanskrit Terms	English Equivalents
1. *ghana-koṇa*	Solid angle.
2. *ghana-kṣetra*	Solid figure.
i) *aṣṭaphalakaghanakṣetra*	Octahedron.
ii) *cheditaghanakṣetra*	Prism.
iii) *ghanahastakṣetra*	Parallelepiped.
iv) *gola-kṣetra*	Sphere.
v) *samabhujadvādaśaphalakakṣetra*	A dodecahedron.
vi) *samānāntara-dharātala-ghanakṣetra*	Parallelepiped.
vii) *samatalamastakaparidhirupaśaṅkukṣetra*	Cylinder.
viii) *sucīphalakaghanakṣetra*	Pyramid.
3. *mastaka.paridhi*	Altitude.
4. *śaṅku*	Cone, cube.
i) *catuṣphalakaśaṅku*	Tetrahedron.
ii) *trasraphalakaśaṅku*	Pyramid having a triangle as its base.
5. *tala*	Base.

CHAPTER IV

ALGEBRA

Early Indian and Chinese geometrical problems involved algebraical equations and their solutions similar to those of the Greeks who solved many comparatively difficult algebraic problems in a purely geometrical way. While the Greek algebra was developed by Diophantus in his *Arithmatica*, in the 3rd century A. D., the algebra in Babylon was developed much earlier in a more advanced form including problems on cubic and biquadratic equations, as recently shown by Neugebauer and others. One can not help wondering whether this Babylonian algebra could have been transmitted in seminal forms to lay the foundation of Indian and Chinese algebra on the one hand and for the Hellenistic development on the other. During the decay of Western Science in the early Middle Ages, the algebra of the Diophantine period was forgotten and when the great Arab Scientific Movement took place, Arabic algebra very probably derived its inspiration from India rather than from Greece.

In India, the geometrical methods of solving algebraic problems, have been traced to the various *Śulbasūtras* of Baudhāyana, Āpastamba, Kātyāyana, Mānava and a few others. These problems, involving solutions of linear, simultaneous and even indeterminate equations, arose in connection with the construction of different types of sacrificial altars and arrangements for laying bricks into them. In the development of early mathematics, when the symbols for operation began to be used in the computations, a new branch evolved being separated from arithmetic and geometry which is known as algebra. The differentiation of algebra as a distinct branch from mathematics in general took place, from about the time of Brahmagupta (628 A. D.), following the technique of indeterminate analysis (*kuṭṭaka*). In fact, Brahmagupta used the terms *kuṭṭaka*, *kuṭṭa-gaṇita* to singnify algebra. The term *bījagaṇita* meaning science of calculation with elements or unknown quantities (*bīja*) was hinted by Pṛthudakasvāmī (850 A. D.) and used with definition by Bhāskara II (1150 A. D.).

Symbols of Operations

Addition was represented by simple juxtaposition of the terms or placing *yu* (*yu* means *yuta* or addition) in between the terms and subtraction by placing a plus sign (+) after the terms which is to be subtracted or a dot at its top; multiplication by placing the abbreviation *gu* (*guṇa* or *guṇita*) after the last term; division by putting the abbreviation *bhā* (*bhāga* or *bhājita*) in between the divisor and the dividend or by placing the divisor beneath the dividend; square-root by writing *mū* or *ka* (*mūla* or *karaṇī*) respectively after the terms and any one unknown quantity by the term *yāvattāvat* (*Sthānāṅga-Sūtra*), *yadṛcchā*, *vāñchā*, *kāmika* (*Bakhshālī Ms.*) *gulikā* (Āryabhaṭa I) and *avyakta* (Brahmagupta, Śrīpati and Bhāskara II). Representation of several unknown terms involved in an equation was made by Brahmagupta (628 A. D.) by using abbreviation of names of several colours like, *kā* (for *kālaka* meaning black), *nī* (for *nīlaka* meaning blue) etc. For sign of equality, the terms *dṛśya* (visible) sometimes *rūpa* with used.

Equations (*samakaraṇa, samīkaraṇa, sadṛśīkaraṇa* etc.) seem to have been classified in the *Sthānāṅgasūtra* (c. 300 B. C.) according to the powers of unknown quantity e.g. *yāvat-tāvat* (simple), *varga* (quadratic), *ghana* (cubic), *varga-varga* (biquadratic) etc. But such classification was not maintained. Brahmagupta (628 A. D.) gave the following classifications : (1) *eka-varṇa samīkaraṇa*–equations in one unknown comprising linear and quadratic equations, (ii) *aneka-varṇa samīkaraṇa* – equations in many unknowns, (iii) *bhāvita* – equations involving products of unknown. This classification received further elaboration at the hands of Pṛthūdaka-svāmī (860 A. D.) and Bhāskara II (1150 A. D.).

Methods and Operations

This primitive method of solving simple linear equations of the type $ax+b=0$ by subtituting guess values g_1, g_2 etc.[1] was in extensive use among the Arab and European mathematicians of the middle ages. The problem of this type dealt with in the *Sthānāṅgasūtra* using the term *yāvat-tāvat* for the unknown quantities has been discussed thoroughly by Datta[2]. A solution to this problem also occurs in the *Bakhshālī Ms*[3] (4th century A. D.).

1. Smith, D. E. *History of Mathematics,* 2, p. 437–8, Dover Publication.
2. Datta, B. "The Jaina School of Mathematics', *Bulletin of the Calcutta Mathematical Society,* 21, p. 122, 1929.
3. Datta, B, 'The Bakhshali Mathematics', *Bulletin of the Calcutta Mathematical Society,* 21, p. 31, 1929.

Āryabhaṭa I[1] might have obtained the solution $x = \frac{b-a}{m-n}$ from the relation $m\,x + a = nx + b$ and tackled the problem of inverse operation e.g. $ax = pq$ and $ax \pm b = p$ by giving the solution as $x = \frac{pq}{a}$ and $ax = p \mp b$ respectively and so on. Bhāskara I (600 A. D.), the earlist exponent of Āryabhaṭa I has used a technical term, *yāvakaraṇa* in his *Āryabhaṭiyabhāṣya.* The word *yāva* (*yu* + *ghañ*) means 'to mix' or 'to separate'. It is also synonymous with *varga* or *kṛti.* The operation on *yāva* has been preserved by Pṛthudaka (850 A. D.) in his commentary on the *Brāhmasphuṭasiddhānta* as such

yāva 0 *yā* 10 *rū* $\dot{8}$.

yāva 1 *yā* 0 *rū* 1. Here *yā* = *yāvattāvat* = unknown = x (say), *yāva* = square of the unknown quantity, *rū* = absolute number. The above can be written as $0.x^2 + 10.x - 8 = 1.\ x^2 + 0.x + 1$, or, $10x - 8 = x^2 + 1$.

The word *yāva* having the idea 'to mix' or 'to separate' has some affinity with that of *aljabr.* The process on the operations (*al-Jabr*) and equations (*al-Muqābala*) has been used by al-Khwārizmī (c. 825 A. D.) in his work *Hīṣāb al-Jabr w'al Muqābalah* like,

The process of *al-Jabr* $\quad bx + 2q = x^2 + bx - q$

or $bx + 2q + q = x^2 + bx$

The process of *al-Muqābalāh*

$$bx + 2q = x^2 + bx - q \text{ or } 3q = x^2$$

Both these process contain the idea of inverse operation similar to Hindu method.

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1 *Āryabhaṭiya,* v. 30, 28.

Quadratic Equation

The *Śulbasūtras* contain problems involving quadratic equations of the type : $ax^2 = c$, $ax^2 + bx = c$ but give no solution. The *Bakhshālī Ms.* gives the solution of a problem in the form which reduces to

$$x = \frac{\sqrt{b^2 - 4ac} - b}{2a}$$

Āryabhaṭa I (499 A. D.) and Brahmagupta (628 A. D.) clearly indicate their knowledge of quadratic equations and their solutions. In connection with an interest prob'em, Āryabhaṭa I gave the solution which in symoo s runs as follows :

$$x = \frac{-p + \sqrt{p^2 + 4tpq}}{2t}$$

where p = principal, t = time, q = sum of interest on principal and interest on interest in time t; x = interest on principal in unit time.

A similar quadratic solution arising out of an interest problem is given by Brahmagupta. He has also given a solution of the problem of quadratic equations in connection with the determination of the number of terms (n) in an A. P.

The method of transforming the left hand side of the quadratic equation $ax^2 + bx = c$ into a whole square by multiplying both sides by 4a, adding b^2 and taking the square root, is given by Śrīdhara (c. 991 A.D.) in his algebra which is lost but preserved in the quotations of Bhāskara II (c. 1150 A. D.), Jñānarāja and Sūryadāsa Datta and Singh discussed in detail all these

Indian methods of solving the quadratic equation in their work, *History of Hindu Mathematics*.[1]

Progressive Series

The usual term for the series is *śreḍhī*. This term is used in almost all mathematical literatures from 5th century A. D. onwards. The other terms used in the *Bakhshālī Ms*.[2] are *saṃkalita*, *varga*[3], *rupona*[4]. Another peculiar term found in the *Bakhshālī Ms*. is *pārtha*. In the *Bṛhaddhārā*[5], the term *dhārā* (ladder) is used. The initial term in an A. P. is *ādi*, *mukha*, *badan* or *baktra* al meaning face. *Antya* or *anta* used to denote the last term; *caya*, *pracaya* or *uttara* stands for the common difference, *gaccha* and *pada* denotes the number of terms, the middle term is *madhya* ; the sum of progression is *sarvadhana*, *śreḍhīphala*, *guṇita* or *citi*.

The term used for the sum of G. P. in the *Triloka-sāra*[6] (978 A. D.) is *guṇa–gaṇita* or *guṇa–saṃkalita*. The terms *śreḍhī*, *ādi*, *gaccha*, *uttara*, *guṇita* were standar-

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1. Datta, B. and *Singh, A. N. History of Hindu Mathematics*, 2, p. 59–87, Lahore, 1938.
2. Folio 23 Recto; folio 4 verso etc.
3. Here it means 'group'.
4. Hoernlė (*Indian Antiquary*, **17**, p. 47) and Kaye (vide his edition of *Bakhshālī Ms*,, p. 73) think that the origin of the term lies in the fact that 'the rule in question began with the term *rupona* which corresponds to the (t–1) of the formula'.
5. The *Bṛhaddhārā-Parikarma*, a treatise on series is extant now. Certain portion of the work appears in the *Triloka-sāra* of Nemicandra.
6. Rule 231,

dised long ago in the Jain canons (2nd century A. D.).

In Indian treatises, three types of series occur viz. arithmetic, complex and geometric series. The classification appeared late in the 9th century A. D. in the *Gaṇitasārasaṃgraha* of Mahāvīra (850 A. D.).

Arithmetic Progression : Indian interest in the series began quite early in the Vedic age. A large number of Vedic series are mentioned in the *Taittirīya Saṃhitā*[1], *Vājasaneya Saṃhitā*[2] and *Pañcaviṃsa Brāhmaṇa*[3]. Some of the few such series are found in the *Bṛhaddevatā*[4], *Śatapatha Brāhmaṇa*[5], *Baudhāyana Śulba*[6], Buddhist work *Dīgha Nikāya*[7] and a Jain work *Antagoḍa Dasāo*[8]. The general rule for the summation of an A. P. : $S = a + (a+b) + (a+2b) + \ldots\ldots + [a + (n-1)b]$ where S denotes the sum, a = first term, b = common difference and n = number of terms, is found in the *Bakhshālī Ms*[9] as follows : $S = n\left[a + (n-1)\frac{b}{2}\right]$. Āryabhaṭa I[10] (476 A. D.) gave the above form as well as the form : $S = \frac{n}{2}(a+b)$. The same rule appears

1. 7.2. 11–17.
2. 18. 24.25.
3. 18.3.
4. Vide Macdonall's English translation, 1904.
5. 10.5.4.7.
6. Datta, B. *The Science of the Śulba,* Calcutta University, p. 217-18, 1932.
7. David, T. W. R. *Dialogue of the Buddha*, 3, p. 70–72, 1928.
8. Ed. by L. D. Barnett, p. 102-6, 1907.
9. *Bulletin of the Calcutta Mathematical Society*, 21, p. 1–60.
10. *Āryabhaṭīya-gaṇita*, v. 19.

in the works of Brahmagupta[1] (628 A. D.), Mahāvīra[2] (850 A. D.), Śrīdhara[3], Nemicandra[4] and Bhāskara II[5] (1150 A. D.). Āryabhaṭa I[6] and all the above writers with the exception of Nemicandra gave also the sum of the following particular case :

$$1+2+3+\ldots\ldots+n = \frac{n}{2}(n+1).$$

Āryabhaṭa I calculated the value of unknown number of terms n for an A. P. series when its sum S, first term a, and commn difference b, are known with the help of quadratic equation as follows :

$$n = \frac{1}{2}\left[\frac{\sqrt{8bS + (2a-b)^2} - 2a}{b} + 1\right]$$

The same was also given by Brahmagupta[7] (628 A. D.), Mahāvīra[8] (850 A. D.) and Bhāskara II[9] (1150 A. D.).

Complex series : (a) Āryabhaṭa I[10] (499 A. D.), Brahmagupta[11], Mahāvīra[12] and also Bhāskara II[13]

1. Colebrooke, H. T, *Hindu Algebra*, p. 290.
2. *Gaṇitasārasaṃgraha*, ch. 2, sl. 61, 62.
3. *Triśatikā*, Sudhākara's edition, rule 39.
4. *Trilokasāra*, rule 164.
5. Colebrooke's *Hindu Algebra*, p. 53.
6. *Āryabhaṭīya-gaṇita*, v. 22; *Brāhmasphuṭasiddhānta* of Brahmagupta, ch. 12, v. 20.
7. Vide Colebrooke's *Hindu Algebra*, p. 291.
8. *Gaṇitasārasaṃgraha*, ch. 2, v. 69.
9. *Līlāvatī*, rule 12 vide H. C. Banerjee's edition .
10. *Āryabhaṭīya-gaṇita*, v. 22, vide Clark's ed.
11. Vide Colebrooke's *Hindu Algebra*, p. 291.
12. *Gaṇitasārasaṃgraha*, ch. 6, v. 296, 301.
13. Vide Colebrooke's *Hindu Algebra*, p. 53.

gave the summation of the series of squares and cubes of n natural numbers as follows :

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n\,(n+1)}{2}\right]^2$$

The formula for the sum of the cubes of the n natural numbers appears in *Codex Arcerianus*[1] (6th century A. D.). The Arabic Scholar āl-Karkhī[2] (c. 1020 A. D.) gave and supplied proofs to the summation to the above two series.

Reference may be made here that the Jain mathematician Mahāvīra (850 A. D.) also gave the results of the generalised (algebraic) summation of the square and cube of the series a, a + b, a + 2b,……corresponding to Āryabhaṭa I's work in natural number as follows :

$$a^2 + (a+b)^2 + (a+2b)^2 + \ldots\ldots\ldots \text{to n terms}$$

$$= n\left[\left\{\frac{(2n-1)\,b^2}{b} + ab\right\}(n-1) + a^2\right]$$

and $a^3 + (a+b)^3 + (a+2b)^3 + \ldots\ldots$ to n terms

$$= S^2 b + Sa\,(a-b) \text{ if } a > b$$

$$= S^2 b - Sa\,(a-b) \text{ if } a < b$$

where $S = a + (a+b) + (a+2b) + \ldots\ldots$ to n terms.

(b) Āryabhaṭa I[3] (476 A. D.) gave the summation of the sums of n times the 1st term of n natural numbers, (n – 1) times the 2nd term of n natural

1. Smith, D. E. *History of Hindu Mathematics*, 2, p. 504, Dover Publication.
2. Cajori, F. *A History of Mathematics*, 2nd revised enlarged edition, p. 100.
3. *Āryabhaṭiya-gaṇita*, v. 21.

numbers, $(n-2)$ times the 3rd term of natural numbers......etc. Thus if

$$S_1 = n.1$$
$$S_2 = (n-1).2$$
$$S_3 = (n-2).3$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$S_n = [\, n-(n-1) \,].n$$

then according to Āryabhaṭa I,

$$S_1 + S_3 + \ldots \ldots + S_n$$
$$= \sum_{n=1}^{n} S_n$$
$$= n.1 + (n-1).2 + (n-2).3 + \ldots\ldots + [n-(n-1)].n$$
$$= 1 + (1+2) + (1+2+3) + \ldots + (1+2+\ldots+n)$$
$$= \frac{n(n+1)(n+2)}{6}$$

Nārāyaṇa (1356 A. D.) calls this as repeated sum or *Vārasaṃkalita.* According to Nārāyaṇa,

Vārasaṃkalita of 1st order of n natural numbers

$$= 1 + 2 + 3 + \ldots\ldots + n$$
$$= \frac{n(n+1)}{2}$$

Vārasaṃkalita of 2nd order of n natural numbers

$$= 1 + (1+2) + (1+2+3) + \ldots n \text{ terms}$$
$$= \sum \frac{n(n+1)}{2} = \frac{n(n+1)\,(n+2)}{6}$$

Āryabhaṭa I gave the result on'y upto *Vārasaṃkalita* of 2nd order, but Nārāyaṇa proceeded further.

His result for the *Vārasaṃkalita* of rth order of a natural numbers[1] is :

$$\frac{n(n+1)\,(n+2)\ldots\ldots(n+r)}{1.2.3.4\ldots\ldots(r+1)}$$

$$=\frac{n(n+1)\,(n+2)\ldots\ldots(n+r)}{(r+1)\,!}$$

In a similar manner Mahāvīra[2] (850 A. D.) also has given an incorrect result of *Vārasaṃkalita* of 2nd order using the series a, a+b, a+2b, ……in place of the series in n natural numbers. The result is

$$\frac{n}{2}\left[\frac{(2n-1)b^2}{6}+\frac{b}{2}+ab\right](n+1)+a(a+1)$$

The correct result is $\frac{n(n+1)}{2}\left[a+\frac{b(n-1)}{3}\right]$

Nārāyaṇa[3] calculated the *Vārasaṃkalita* of rth order by employing the general algebraic progressive series like a, a+b, a+2b… . The given result is :

$$a.\,{}^{n-1}S_r.\,\frac{r+1}{n-1}+b.\,{}^{n-1}S_r$$

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1. *ekādhikavāramitāḥ padādirupottarāḥ pṛthaktemʼśāḥ* ।
ekādyekacayahaśāstaddhāto vārasaṃkalitam ॥
(*Gaṇitakaumudī*, Part II, p. 411).

"The number beginning with the number of terms in the series increasing by one and equal in number to one more than the number representing the order of summation separately form the numerators. The corresponding denominators are the natural number beginning with the product of these (fractions) is the *Vārasaṃkalita*".

2. *Gaṇitasārasaṃgraha*, ch. 6, rule 305½.

3. *Gaṇitakaumudī*, Part I , p. 124. Vide editor's note. For edition, see *General Bibliography*.

when a, b and n are the 1st term, common difference and number of terms of an A. P. Nārāyaṇa represented the parallel sides of a trapezia by $a-\frac{b}{2}$ and $nb+a-\frac{b}{2}$ and the altitude by n. Arithmetic Progression with sum = 0 are diagrammatically represented by joining two equal isosceles inverted triangles with their vertices.

(c) In the *Bakhshālī Ms.*[1] certain elementary cases of complex series without sum are mentioned. These series are classified into two types viz. (i) *yutivarga-krama* and (ii) *yutagaṇita-yutakrama.* Mahāvīra[2] (850 A. D.) gave the sum of the following complex series which may be considered as one of his highest contribution in this branch :

$$a+(ar\pm m)+[\,(ar\pm m)r\pm m\,]+$$
$$[\,\{(ar\pm m)r\pm m\}\,r\pm m\,]+\ldots\text{to n terms}$$
$$=S'\pm\frac{\left(\frac{S'}{a}-n\right)m}{r-1}$$

where $S'=a+ar+ar^2+\ldots$ to n terms.

Geometrical Progression : Earliest expression of an idea representing G. P. occurs in the *Chandaḥ-sūtra*[3] (rule of metres) of Piṅgala (c. 200 B. C.) where the series : 1, 2^1, 2^2, 2^3...is found. Mahāvīra[4] (850

1. Datta, B. 'The Bakhshālī Mathematics', *Bulletin of the Calcutta Mathematical Society*, **21**, p. 30–31, 1929.
2. *Gaṇitasārasaṃgraha*, ch. 2, rule 314.
3. *Chandaḥ-sūtra* of Piṅgala, rules 8, 28, 29, 30, 31, 32.
4. *Gaṇitasārasaṃgraha*, ch. 2, rules 94 and 93.

A. D.) gives the generalised result of the sum of the n terms of the G. P. as follows :

$$a + ar + ar^2 + \ldots\ldots \text{to n terms} = \frac{a(r^n - 1)}{r - 1}$$

This formula reappears also in the works of Pṛthudakasvāmī[1] (864 A.D.) and Nemicandra[2] (978 A. D.).

Mahāvīra gave also the form $S = \frac{rar^{n-1} - a}{r - 1}$

The rule appears amongst the Arabians in the work of āl-Bīrūnī's *India* in connection with a chess-board problem which he might have learnt during his tour in India[3].

From the above, Mahāvīra deduced three expressions[4] also for finding the common ratio (r), first term (a) and number of terms (n).

Permutation and Combination

Smith[5] opined that "the Hindus seem to have given no attention to permutation and combination until Bhāskara II took it up in his *Līlāvatī*". Chakravarty[6] has shown that the subject of permutations and combinations is a noteworthy contribution of the Indians and its interest originated before the Christian

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1. Colebrooke, *Hindu Algebra*. p. 291.
2. *Trilokasāra*, rule 231.
3. Smith, D. E. *Ibid.*, 2, p. 502, Dover Publication.
4. *Gaṇitasārasaṃgraha*, ch. 2, rules 97, 98 and 101.
5. Smith, D. E. *History of Mathematics*, 2, p. 525, Dover Publication.
6. Chakravarty, G. 'Growth and Development of Permutations and Combinations in India', *Bulletin of the Calcutta Mathematical Society*, 24, p. 79-88, 1932.

era. A summary is also given by Srinivasienger.[1] In the early Jain canonical literature, permutation was termed *Vikalpa-gaṇita* and combination *Bhaṅga.* The Jain *Bhagavatī-sūtra* (300 B. C.) calculated the number of combinations of n fundamental categories taken two at a time, three at a time, and so on.[2] Suśruta (200 B. C.) correctly gave the sum of the combinations of six tastes taken one at a time, two at a time etc. upto all at a time.[3] Varāhamihira[4] (505 A. D.) has stated that "an immense number of perfumes can be made from sixteen substances taken in one, two, three or four proportions and has correctly given the number of perfumes resulting from 16 ingredients (being mixed in all proportions) as 174720". The results given agree with the modern formulae though there is no mention of any such formula in his text.

The great astronomer and encyclopaedist Varāhamihira applied the same principle in his astrological work *Bṛhajjātaka* in connection with planetary conjunctions.[5]

Mahāvīra[6] (c. 850 A.D.) first gave the general formula for ^nc_r. Bhāskara II (c. 1150 A. D.) gave

1. Srinivasiengar, C. N. *The History of Ancient Indian Mathematics*, p. 26-28, 73, 83-84, Calcutta, 1967.
2. Sūtra, 314; vide also *Bulletin of the Calcutta Mathematical Society*, 21, p. 133-136, 1929.
3. *Suśrutasaṃhitā*, ch. 63. (*Rasabhedavikalpādhyāya*).
4. *Bṛhatsamhitā*, ch. 77, sl. 13, 14, 17 (for the translation vide Kern's work).
5. *Bṛhajjātaka*, rule 3.
6. *Gaṇitasārasaṃgraha*, ch. 3, v. 218.

the name *ankapāśa* to the subject of permutation and combination. He also gave the general formulae for ^nc_r and ^np_r. Bhāskara II made some other valuab'e contributions to the subject of permutation and combination. He gave the rules[1] to find :

i) the variation of the number of permutaticn of a number consisting of n digits of which p number of digits are to be alike and q also are to be alike, the result is $\frac{n!}{p!q!}$;

ii) the variation of numbers each of which contains n digits with a sum equal to S, zero should not be counted as a digit. The result of the problem is given by : $^{S-1}c_{n-1}$. This result holds good when $9+n>S$.

Bhāskara II gave no proof. Banerjee supplied the plausible proofs to the latter.[2]

Pascal Triangle and the Binomial Theorem

The credit of expansion of Binomial theorem :

$(a+b)^n = a^n + {}^nc_1\, a^{n-1}b + {}^nc_2\, a^{n-2}\, b^2 + \dots\dots\dots + b^n$ in its simplest form i.e., when n is a positive integer, goes to India and it is the Indian scho!ar Halāyudha (10th century A. D.) who established it much before the Chinese and European scholars could do anything about it[3] It is based on a concept which developed in India in association with the problem of making verses of different metres (*chandas*) in a poetical com-

1. *Līlāvatī*, rules 267, 272, also compare with 274.
2. Vide Haran Chandra Banerjee's edition of *Līlāvatī*, p. 197-200.
3. Bag, A. K. 'Binomial Theorem in Ancient India', *Indian Journal of History of Science*, 1, No. 1, p. 68–74, 1966.

position. The *Varṇasaṅgīta* (the music of sound variations) of the Vedic and post Vedic composers depended only on the variation of two sounds namely *guru* (long) and *laghu* (short) sounds. Piṅgala (200 B. C.) in his *Chandaḥsūtra*[1] described clearly a rule of finding the variation of sounds of syllables in a particular metre. For example, in a three syllabic *madhyā chanda* based on *guru* and *laghu*, the variations of *guru* and *laghu* will be — three *guru* sounds will occur once, two *guru* and one *laghu* sound will occur thrice, one *guru* and two *laghu* sounds occur thrice, three *laghu* sounds will occur once. If we take *guru* as *a* and *laghu* as *b*, then it gives $(a+b) = 1.a^3 + 3a^2b + 3ab^2 + 1.b^3$. Similarly for a four syllabic *Pratiṭhā chanda*, it uses $(a+b)^4 = 1.a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1.b^4$. The technique of finding the number of variation of sounds was actually practised by Piṅgala and other Vedic composers[2] to detect the quality as well as shortcomings of the metres.

Halāyudha (10-11th century A.D.) in his commentary on the Piṅgala *Chandaḥsutra*[3] (ch. 8, verse 34) introduced a *meruprastāra* (pyramidal scheme) technique with the coefficients of the different variations in the expansion of $(a+b)^1$, $(a+b)^2$, $(a+b)^3$........to minimise the trouble of computing the different layers which run thus :

"Draw a square on the top, two squares are drawn below (side by side) so that half of each is extended

1. *Chandaḥ sūtra* of Piṅgalācārya, edited with the commentary of Halāyudha by Sitanath Sarman, Calcutta.
2. Vide *Jayadaman*, edited by Velāṅkar.
3. Ch. 8, Verse 34.

on either side. Below it three squares, below it (*again*) four squares are drawn and the process is repeated till the desired pyramid is attained. In the (topmost) first square the symbol for one is to be marked. Then in each of the two squares of the second line figure one is to be marked. Then in the third line figure one is to be placed on each of the two extreme squares. In the middle square (of the third line) the sum of the figures in the two squares immediately above is to be placed ; this is the meaning of the term *pūrṇa*. In the fourth line one is to be placed in each of the two extreme squares. In each of the middle squares, the sum of the figures in the two squares immediately above, that is, three is placed. Subsequent squares are filled in this way. Thus the second line gives the expansion of combinations of (short and long sounds forming) in a one syllabic metre ; the third line the same for two syllables, the fourth line for three syllables *meruprastāra* and so on." [vide 1 Fig. 17 (a), 17 (b) and 17 (c).]

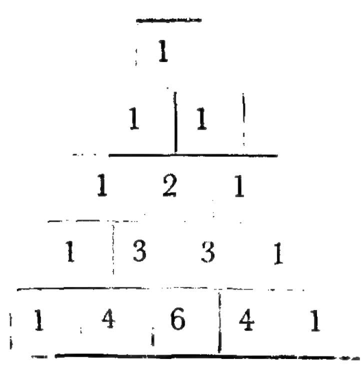

Fig. 17 (a).

$$\begin{array}{ccc} 1 & & \\ 1(a)\ 1(b) & \rightarrow & (a+b)^1 \\ 1(a)\ 2(ab)\ 1(b^2) & \rightarrow & (a+b)^2 \\ 1(a^3)\ 3(a^2b)\ 3(ab^2)\ 1(b^3) & \rightarrow & (a+b)^3 \\ 1(a^4)\ 4(a^3b)\ 6(a^2b^2)\ 4(ab)\ 1(b^4) & \rightarrow & (a+b)^4 \\ \ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots \end{array}$$

Fig. 17 (b).

In modern notation this gives to :

$$\begin{array}{ccc} 1 & & \\ {}^1c_0\ {}^1c_1 & \rightarrow & (a+b)^1 \\ {}^2c_0\ {}^2c_1\ {}^2c_2 & \rightarrow & (a+b)^2 \\ {}^3c_0\ {}^3c_1\ {}^3c_2\ {}^3c_3 & \rightarrow & (a+b)^3 \\ {}^4c_0\ {}^4c_1\ {}^4c_2\ {}^4c_3\ {}^4c_4 & \rightarrow & (a+b)^4 \\ \ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots \end{array}$$

Fig. 17 (c).

In this way, the general expansion of

$(a+b)^n = a^n + n_{c_1}\, a^{n-1}\, b + \ldots\ldots\ldots\ldots + b^n$ was readily obtained for a metre of n syllables.

The exactly same technique of arriving at the triangular array appeared in China (1303 A.D.) in the work *Ssu Yuan Yü Chien*[1] ('the Old Method Chart of Seven Multiplying Squares') of Chu Shih Chieh and

1. Mikamī, Y. *Mathematics in China and Japan*, New York, p. 90; Smith, D. E. *History of Mathematics*, 2, p. 508, Dover Publication; Needham, J. *Science and Civilisation in China*, 3, p. 134.

in Europe in the publication of Pascal's *Traitè du Triangle Arithmetic* (1665 A.D.) The Triangle, however, appeared in Europe more than hundred years before the *Traitè* on the title page of the arithmetic of Apianus (1527 A.D.), and in the work of Stifel (1544 A D.), Scheubel (1545 A.D.), Tartaglia (1556 A.D.), Bombelli (1572 A.D.) and others[1], but about four centuries later before it actually appeared in India.

Indeterminate Equations of the First Degree

The subject of indeterminate analysis (first degree) is designated in Indian mathematics by the term *kuṭṭaka*. *Kuṭṭ* means 'to break' or 'to pulverize'. The name has been given on account of the process of continued division that is adopted for the solution. Indian interest in the indeterminate problems is first found in the *Śulbasūtras*, Detailed rules of solutions of the type : $by = ax \pm c$ were given by scholars from Āryabhaṭa I (476 A. D.) onwards Colebrooke[2], Chasles[3] and Sen[4] gave credit to Indians for attaining a general solution of the problem. Some scholars favoured Greeks and the Chinese for the origin. They are of opinion that Indians got inspiration in this sub-

1. Smith, D. E. *History of Mathematics*, 2, p. 508-11, Dover Publication.
2. Colebrooke, H. T. *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupte*, and *Bhāscara*, London, 1817.
3. Chasles, M. *Apercu historique sur l'origine et le developpement des methodes en geometric*, 2e ed. Paris, 1875.
4. Sen, S. N. 'Study of Indeterminate analysis in Ancient India' *Bulletin of History of science Congress*, held at Ithaca p. 493-97, 1962.

ject from the foreign countries. In the foregoing discussion it can be seen that while the career of indeterminate (first degree) is doubtful in other cultural areas, in India it is based on strong and solid foundation.

The problems of altar construction in the *Śulba-sūtras* necessitated the evolution of simultaneous indeterminate equations of the type :

i) $x+y=21$

$$\frac{x}{p^2}+\frac{y}{q^2}=1$$

ii) $x+y+z+u=200$

$$\frac{x}{m}+\frac{y}{n}+\frac{z}{p}+\frac{u}{q}=7\tfrac{1}{2} \quad \text{etc.}$$

A few arbitrary solutions were given by Baudhāyana (c. 600 B. C.), Āpastamba (c. 550 B. C.), and their commentators. The rational solutions of such equations were given by Datta[1] and Kamalamma.[2]

Āryabhaṭa I (b. 476 A. D.) first gave a rule for obtaining the general solution of the problem $by=ax \pm c$. His rule was obviously intended for the solution of the following problems :

(1) To find the number (N), which when divided by a given number (a) will leave a remainder (r_1) and when divided by another given number (b) will leave a remainder (r_2).

(2) To find a number (N) which being divided severally by the given numbers $a_1, a_2, a_3, \ldots\ldots a_n$ leaves remainders as $r_1, r_2, r_3 \ldots\ldots r_n$ respectively.

1. Datta, B. *The Science in the Śulba*, p. 180-86, Calcutta University, 1932.
2. Bulletin of the Calcutta Math. Society (BCMS), **40**, p. 140-44, 1948.

In modern notation, this can be written as,

(1) $N = ax + r_1 = by + r_2$

(2) $N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \dots\dots$

$$= a_nx_n + r_n.$$

Nature of Solution (Āryabhaṭa I)

Āryabhaṭa I keeps $c = r_1 - r_2$ always positive, consequently his rule is directed to solve an equation of the form :

i) $y = \frac{ax + c}{b}$ if $r_1 > r_2$ and

ii) $x = \frac{by + c}{a}$ if $r_2 > r_1$.

Where a, b = divisors, ***bhāgahāra, bhājaka, cheda*** etc.

r_1, r_2 = remainders, ***agra, śeṣa*** etc.

Āryabhaṭa I's rule[1] :

adhikāgra bhāgahāraṃ chindyādūnāgra bhāgahāreṇa ।
śeṣaparasparabhaktaṃ matiguṇamagrāntare kṣiptam ॥
adhauparigunitamantyayugūnāgracchedabhājite śeṣam ।
adhikāgracchedaguṇaṃ dvicchedāgramadhikāgrayutaṃ ॥

English Translaton :

We follow Datta who based his translation on that of Bhāskara I (600 A. D.), the first commentator on Āryabhaṭa I's work. The translation runs as follows : "Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder, being mutually divided, the last residue should be multiplied by an integer at our choice such that the product on being added to (if the number of quotients in the division process is odd) or subtracted (if the number of quotients is even) by the difference of the remainders, will be exactly divisible by the

1. *Āryabhaṭiya*, Gaṇita, vss. 32–33.

penultimate remainder. Place the quotients of the mutual division successively one below the other in a column, below them the optional multiplier and below it the quotient just obtained. The penultimate is multiplied by the one just above it and added to that just below it. Repeating this process, divide the last number obtained by the divisor corresponding to the smaller remainder. Then multiply the residue by the divisor corresponding to the greater remainder and add greater remainder. The result will be a number corresponding to the divisors.'[1]

Explanation of the Rule :

i) Solution of $y = \frac{ax+c}{b}$ ....(1) in positive integers when a and b are prime to each other.

Repeated divisions following H. C. F. process may be placed as follows :—

$b)\ a\ (a_1$
$\quad ba_1$
$\quad r_1)\ b\ (a_2$
$\qquad r_1a_2$
$\qquad r_2)\ r_1\ (a_3$
$\qquad\quad r_2a_3$
$\qquad\quad r_3)\ r_2\ (a_4$
$\qquad\qquad r_3a_4$
$\qquad\qquad r_4$
...
...
...
...
$r_{n-1})\ r_{n-2}\ (a_n$
$\qquad r_{n-1}a_n$
$\qquad r_n$

$a = ba_1 + r_1$
$b = r_1a_2 + r_2$
$r_1 = r_2a_3 + r_3$
$r_2 = r_3a_4 + r_4$
and so on.

1. *Bulletin of the Calcutta Mathematical Society*, 24,p.19–36,1933.

then $\frac{a}{b} = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4+} \cdots\cdots \frac{1}{a_{n-1}+}\ \frac{1}{a_n}$

Let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots\ldots\ldots\ldots \frac{p_n}{q_n}$ be the successive convergents of $\frac{a}{b}$, then :

$$\frac{p_1}{q_1} = a_1$$

$$\frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_1 a_2 + 1}{a_2}$$

$$\frac{p_3}{q_3} = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3}$$

$$= a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}$$

$$= a_1 + \frac{a_3}{a_2 a_3 + 1} = \frac{a_1(a_2 a_3 + 1) + a_3}{a_2 a_3 + 1}$$

$$\frac{p_4}{q_4} = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4}$$

$$= a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}$$

$$= a_1 + \cfrac{1}{a_2 + \cfrac{a_4}{a_3 a_4 + 1}} = a_1 + \frac{a_3 a_4 + 1}{a_2(a_3 a_4 + 1) + a_4}$$

$$= \frac{a_1\{a_2(a_3 a_4 + 1) + a_4\} + a_3 a_4 + 1}{a_2(a_3 a_4 + 1) + a_4}$$

and so on, and the last convergent $\frac{p_n}{q_n} = \frac{a}{b}$

Evidently,

$p_n \cdot q_{n-1} - q_n \cdot p_{n-1} = \pm 1$ according as $n =$ even or odd.

This result was perhaps known to Āryabhaṭa I for he always used the result $p_n . v - q_n . u = -c$ (whether n is even or odd), the two numbers u and v were obtained by following an ingenious technique discussed later.

Case I : When n = even i.e. the number of partial quotients for finding u and v is odd (vide Table I)

The cryptic Sanskrit terms *agrāntare kṣiptam* used by Āryabhaṭa I in the rule has been explained by his commentator Bhāskara I (c.600 A.D.) in his *Āryabhaṭīya-bhāṣya* as *sameṣu kṣiptam, viṣameṣu, śodhyam,* i.e. add (the *kṣepa* quantity, c) when n = even, and subtract when n = odd." Since the ratio $\frac{u}{v}$ was always used to determind the (n – 1) th convergent, the *kṣepa* quantity c was to be added or subtracted according as the number of partial quotents arranged in Table I & II are odd and even. Suppose the process of division was stopped at remainder r_3

$$\text{then } \frac{r_3 t + c}{r_2} = q_1 \text{ say, where } t = \frac{r_2 q_1 - c}{r_3}$$

The quotients a_1, a_2, a_3 with *mati* t, and final quotient q_1 are placed one after another in the first column (vide Table I) and the final numbers u and v were obtained as follows :—

Table I

a_1	a_1	a_1	$a_1 \; v + \alpha_1 =$
a_2	a_2	$a_2 \; \alpha_1 + t = v$, say	u, say
a_3	$a_3 \; t + q_1 = \alpha_1$, say	α_1	
t	t		
q_1			

Now $\alpha_1 = a_3 t + q_1$

$$= a_3\left(\frac{r_2\ q_1 - c}{r_3}\right) + q_1 \qquad \because t = \frac{r_2\ q_1 - c}{r_3}$$

$$= \frac{q_1\ (r_2\ a_3 + r_3) - a_3\ c}{r_3}$$

$$= \frac{q_1\ r_1 - a_3\ c}{r_3} \qquad \text{(p. 196)}$$

$v = a_2 \alpha_1 + t$

$$= a_2 \left\{\frac{q_1\ r_1 - a_3\ c}{r_3}\right\} + \frac{r_2\ q_1 - c}{r_3}$$

$$= \frac{q_1\ (a_2 r_1 + r_2) - c\ (a_2 a_3 + 1)}{r_3}$$

$$= \frac{q_1 b - c\ (a_2 a_3 + 1)}{r_3} \qquad \text{(p. 196)}$$

$u = a_1\ v + \alpha_1$

$$= a_1 \left\{\frac{q_1\ b - c\ (a_2\ a_3 + 1)}{r_3}\right\} + \frac{q_1\ r_1 - a_3\ c}{r_3}$$

$$= \frac{q_1\ (a_1\ b + r_1) - c\ \{a_1\ (a_2\ a_3 + 1) + a_3\}}{r_3}$$

$$= \frac{q_1\ a - c\ \{a_1\ (a_2\ a_3 + 1) + a_3\}}{r_3}$$

$$\frac{u}{v} = \frac{q_1\ a - c\ \{a_1\ (a_2\ a_3 + 1) + a_3\}}{q_1\ b - c\ (a_2\ a_3 + 1)}$$

$$= \frac{q_1\ a - c p_3}{q_1\ b - c q_3}$$

If there be $(n-1)$ number of partial quotients in the Table I, then

$$\frac{u}{v} = \frac{q_1\ a - p_{n-1} \cdot c}{q_1\ b - q_{n-1} \cdot c}$$

The last n th convergent is always $\frac{a}{b}$

hence $\frac{p_n}{q_n} = \frac{a}{b}.$

Āryabhaṭa I knew,

$$p_n.\, v - q_n.\, u = -c \; [\because n = \text{even}] \; \ldots \quad \ldots \quad (2)$$

For. $p_n\,(q_1 b - c q_{n-1}) - q_n\,(q_1\, a - c p_{n-1})$

$$= -c(p_n\, q_{n-1} - q_n\, p_{n-1}) \; (\because p_n = a,\; q_n = b)$$

$$= -c$$

Case II : When n = odd i.e, the number of partial qutient[1] *used for finding u and v is even.*

Suppose the repeated division was stopped at the remainder r_4. Following rule, let us put,

$$\frac{r_4 t - c}{r_3} = q_2, \text{ where } t = \frac{r_3\, q_2 + c}{r_4}$$

Now we place below the quotients a_1, a_2, a_3, a_4, then *mati*, t and c and obtain u and v following a process of multiplication and addition as directed in the rule.

Table II

a_1	a_1	a_1	a_1	$a_1 v + \beta_2 = u$ (say)
a_2	a_2	a_3	$a_2\beta_2 + \beta_1 = v$	
a_3	a_2	$a_3\beta_1 + t = \beta_2$	β_2	
a_4	$a_4 t + q_2 = \beta_1$	β_1		
t	t			
q_2				

If we calculate β_1, β_2, v, and u proceeding as before we will find, $\dfrac{u}{v} = \dfrac{q_2 a + c p_4}{q_2 b + c q_4}$

1. Shukla, Kripa Shankar, *Mahābhāskarīya*, edited and translated into English with notes, p. 30–33, Lucknow University, 1960.

If there be $(n-1)$ number of partial quotients in the Table II, then :

$$\frac{u}{v}=\frac{q_2a+cp_{n-1}}{q_2b+cq_{n-1}}$$

Āryabhaṭa I knew that :

$$p_n \cdot v-q_n \cdot u=-c[n=\text{odd}] \quad \dots \quad \dots \quad (3)$$

For $p_n \cdot v-q_n \cdot u$

$$=p_n(q_2b+cq_{n-1})-q_n(q_2a+cp_{n-1})$$
$$=c(p_nq_{n-1}-q_np_{n-1}) \; [\because p_n=a, \; q_n=b]$$
$$=-c[n=\text{odd}]$$

The equations (2) & (3) are identical. This was purposely done by Āryabhaṭa I, by selecting $\frac{r_nt\pm c}{r_{n-1}}=q_1$ or q_2 according as the number of partial quotients in the Table are odd or even (chosen by Āryabhaṭa I.)

From (3), $q_n u=p_n \cdot v+c$

or $\quad bu=av+c \quad \dots \quad \dots \quad \dots \quad \dots(4)$

Evidently $y=u$ and $x=v$

when $\quad u=am+y$, and $v=bm+x$,

the equation (4) reduces to

$$by_1=ax_1+c$$

$\therefore$ $y\equiv y_1$ (mod a) and $x=x_1$ (mod b)

hence (x_1, y_1) is the least solution of (1)

ii) Solution of $x=\frac{by+c}{a}$ in positive integer when a and b are prime to each other.

The method of solution is exactly similar, only select $\frac{r_nt\pm c}{r_{n-1}}=q_1$ or q_2 according as the number of partial quotients are odd or even.

then $x = u = bm + x_1$
or $x \equiv x_1 \pmod{b}$.
and $y = v = am + y_1$
$\equiv y_1 \pmod{a}$

(2) *Solution of linear simultaneous indeterminate equation*

To solve $N = a_1x + r_1 = a_2y + r_2 = a_3z + r_3 = \dots = a_nt + r_n$.

Then the least value of $N = a_1\alpha + r_1$

General value $= a_1 \; a_2n + a_1\alpha + r_1$ where n is any integer. Thus we find that $a_1\alpha + r_1$ is the remainder left on dividing N by the product a_1 and a_2. This is exactly what has been stated by Āryabhaṭa I as follows : "(The result will be) the residue corresponding to the product of the two divisors".

Let $N' = a_1 \; a_2n + (a_1\alpha + r_1) = a_3z + r_3$

then $N' = a_1 \; a_2 \; a_3t + a_1\alpha + r_1$ which shows that N' when divided severally by a_1 a_2 and a_3 leaves as remainder a_1 α and r_3 respectively and repeat the process and so on.

Illustrations :

(1) $45x + 7 = 29y$

Now, 29) 45 (1
29
16) 29 (1
16
13) 16 (1
13
3) 13 (4
12
1

Here no. of quotients omitting the first is odd.

Therefore, $\frac{1 \times 10 - 7}{3} = 1$

	1	1	1	1	143
Now	1	1	1	92	92
	1	1	51	51	
	4	41	41		
	10 (*mati*)	10			
	1				

Now $y = 143 = 45 \times 3 + 8 = 8 \pmod{45}$

and $x = 5 \pmod{29}$......

(2) $N = 12x + 5 = 31y + 7$

To find x, y, and N.

Now $31y + 2 = 12x$

```
12)  31  (2
     24
    ----
     7) 12  (1
         7
       ----
        5)  7  (1
            5
            2
```

Here the no. of quotients excepting the first = even.

Here $\frac{2 \times 4 + 2}{5} = 2$

Table	2	2	2	26	
	1	1	10	10	
	1	6	6		
	4 (*mati*)	4			
	2				

Now $x = 26, \quad y = 10$

$\therefore N = 31y + 7 = 317$

Bhāskara I : Bhāskara I (600 A.D.) stated clearly that a and b should be prime to each other. He put the equation always in the form $\frac{ax - c}{b} = y$ which is same as Āryabhaṭa I's $\frac{by + c}{a} = x$. His rule is similar to that of Āryabhaṭa I and was given in his *Mahāsiddhānta* (ch. 1, v. 41–44). He for the first time used this equation to astronomy to compute the *ahargaṇas* and revolutions of the planets. He gave also the following results :

(i) If (x_1, y_1) be solutions of $ax + 1 = by$, then (cx_1, cy_1) is solution of $ax + c = by$. (*Mahābhāskarīya,* ch. 1, v. 45–46i).

(ii) If $x = \alpha$, $y = \beta$ be a solution of $Ax - C = by$, then $x = \alpha$, $y = m\alpha + \beta$ be a solution of $\frac{ax - C}{b} = y$, where $a = mb + A$. (*Mahābhāskarīya,* ch. 1. v. 47).

(iii) If $cx_1 \equiv x'_1$ (mod b), $cy_1 \equiv y'_1$ (mod a), then $(x'_1\ y'_1)$ is the solution of $ax + C = by$. (*Mahābhāskarīya* ch. 1, v. 50).

Various problems were given by Bhāskara I, a few of which are given here :

(1) "What is that number which being divided by 12, leaves the remainder 5; again when divided by 31 has the remainder 7 ?" (*Āryabhaṭīya bhāṣya*).

Hints : To solve $N = 12x + 5 = 31y + 7$.

Here $N = 317$, $x = 31$, $y = 10$.

(2) "Find what is that number which divided by 8 is known to leave a remainder 5; by 9 a remainder 4; and by 7 a remainder 1". (*Ārya-bhaṭīya bhāṣya*).

Solution : $N = 8x + 5 = 9y + 4 = 7z + 1$ (say).

Let $N' = 8x + 5 = 9y + 4$.

The least value of N is found to be 13 (by the method of pulverizer). This remainder is left by dividing the number N′ by the product $8.9 = 72$. Hence in general $N = 72t + 13$ where t is any arbitrary integer. Again, $N = 72t + 13 = 7z + 1$. Applying the same method again, least value of $N = 85$ which satisfies the three given divisors.

(3) Find a number N which leaves the remainder 1, when divided by 2, 3, 4, 5 or 6 but is exactly divisible by 7". (*Āryabhaṭīya bhāṣya*).

This problem has been treated by Ibn-al-Haitam (c. 1000 A. D.), Leonardo Fibonacci (c. 1202 A. D.) and two writers of the 17th century.

Hints : $N = 2x + 1 = 3y + 1 = 4z + 1 = 5u + 1$

$= 6v + 1 = 7t + 0.$

Here $N = 721$.

(4) "The residue of the revolutions of Saturn is 24, find the *ahargaṇa* and the revolutions performed by Saturn" (*Laghubhāskarīya*, ch. 8, v. 17).

Hints : The revolution number of Saturn = 146564.
No. of civil days in a yuga = 1577917500.
These two figures have a common factor 4.
Hence dividing them by 4, we get 36641

and 394479375 respectively. Therefore one has to solve the equation.

$$36641x - 24 = 394479375y.$$

where x = *ahargaṇa* and y = revolutions performed by Saturn.

Here x = 346688814, y = 32202.

(5) "The mean (position) of the Sun has been observed by me at sun-rise to be in the sign Leo in the middle of the *navaṃśa* Sagittarius.[1] Calculate the *ahargaṇa* according to the (Ārya)bhaṭaśāstra and also the revolutions performed by the Sun since the beginning of Kaliyuga". (*Mahābhāskarīya*, ch. 2, v. 32–33)

Hints : The mean longitude of the Sun = 4 signs 28°20′ = 8900′. The abraded revolution number of the Sun = 576. The abraded number of civil days in a yuga = 210389. Hence the residue[2] of the revolution = 86688.

Now to solve the equation $\frac{576x - 86688}{210389} = y,$

1. One *aṃśa* = 3°20′. Hence middle of the *navamaṃśa* = 3°20′ × 8 + 1°40′ = 28°20′.

2. Residue of the revolutions = (Mean longitude × abraded number of civil days in a yuga) ÷ Number of divisions (in zodiac).

No. of zodiacal divisions would be 12, 360, and 21600 accordingly as the mean longitude is expressed in terms of signs and degrees, signs and degrees and minutes respectively.

where $x = $ *ahargaṇa*, $y =$ no. of revolution performed by the Sun.

Hore $x = 105345$, $y = 288$.

General solution : $x = 210389\alpha + 105345$.

$y = 576\alpha + 288$

Scholars from Brahmagupta to Āryabhaṭa II :

The method was subsequently discussed by Brahmagupta (c. 628 A. D.), Govindasvāmī (c. 850 A. D.) Mahāvīra (850 A. D.), Pṛthudakasvāmī (c. 861 A. D.) and others. The method of solution was further reduced by Āryabhaṭa II (950 A. D.). He continued the mutual division till the remainder becomes 1. Then the table (*valli*) was made with quotients and with 1 and 0 attached at the end as follows.

a_1	a_1	a_1	$a_1\ (a_2 a_3 + 1) \equiv u$ say
a_2	a_2	$a_2 a_3 + 1$	$a_2 a_3 + 1 \equiv v$, say
a_3	$a_3.1 + 0 = a_3$	a_3	
1	1		
0			

$$\text{then } \frac{u}{v} = \frac{a_1 (a_2 a_3 + 1) + a_3}{a_2 a_3 + 1} = \frac{p_3}{q_3}$$

Similarly if there be $(n-1)$ partial quotients in the table, then $\frac{u}{v} = \frac{p_{n-1}}{q_{n-1}}$

Since the mutual division was not continued upto the remainder zero but upto the one i.e., one operation less in the continued fraction.

$$\text{i. e. } \frac{p_n}{q_n} = \frac{a}{b}$$

Āryabhaṭa II, similar to Aryabhaṭa I. knew that $p_n . v - q_n . u = \pm 1$ according as total number of partial quotients are even or odd.

or $a v - b u = \pm 1$ according as the number of partial quotients in the above Table or odd or even.

When the number of partial quotients in the table are even, $a v - b u = -1$ or $b u = a v + 1$, then u and v is the solution of $by = ax + 1$. If $u = at + \beta$, and $v = bt + a$, then (a, β) is the least solution of $by = ax + 1$.

When the number of partial quotients in the table are odd, then $a v - b u = +1$ or $b u = a v - 1$, then (u, v) is a solution of $b y = ax - 1$.

If $u = at + a - \beta$ and $v = bt + b - \alpha$, then the equation becomes $b\beta = a\alpha + 1$ i.e. (α, β) is the least solution of $by = ax + 1$

This gives the soluttion of $by = ax \pm 1$......(1). Then if α and β be the solutions of (1) then $c\alpha$ and $c\beta$ are solution of $by = ax \pm c$.

Here two cases should be considered depending on the number of quotients as follows :

Case 1 : When the number of quotients are even, the method gives the solution of $ax + c = by$.

Case 2 : When the number of quotients are odd, the method gives the solution of $ax - c = by$. Then $x = (b - \alpha) + bt$, $y = (a - \beta) + at$ is the general solution of $ax + c = by$.

Āryabhaṭa II eliminated the common factors among a, b and c and suggested also the following steps when the numbers (a, c) or (b, c) have a common factors (*Mahāsidhānta*, ch. 18, v. 1–2).

i) If $a = ka'$, $c = kc'$ where k is an integer. The equation $ax + c = by$ reduces to $a'x + c' = by'$ where $y' = \frac{y}{k}$. Hence if (x, y) be solutions of the reduced equation, (x, ky') is the solution of $ax + c = by$.

ii) If $c = kc'$, $b = kb'$, the equation $ax + c = by$ reduces to $ax' + c' = b'y$ where $x' = \frac{x}{k}$. If (x, y) be a solution of $ax' + c' = b'y$, then (kx', y) is the solution of $ax + c = by$.

iii) If $a = ka'$, $c = kc'$, again $c' = lc''$ and $b = lb''$, then the equation $ax + c = by$ ... ... (1) becomes $ka'x + klc'' = lb''y$.

$$\text{or} \quad \frac{ka'x}{kl} + \frac{klc''}{kl} = \frac{lb''y}{kl} \quad \text{or} \quad a'.\frac{x}{l} + c'' = b''.\frac{y}{k}$$

$$\text{or} \quad a'X + c'' = b''Y \quad \text{... ... (2) when } X = \frac{x}{l},$$

$$Y = \frac{y}{k}$$

Hence if (x, y) is solution of (2), then (Xl, Yk) is solution of (1).

Bhāskara II (1150 A. D.) simply adopted the method of Āryabhaṭa II i.e. continued the mutual division till the remainder becomes one and prepared the table (*valli*) with quotients along with c and o. This shows that Bhāskara II directly calculated $bu = al \pm c$ (according as the number of partial quotients in the table is odd or even), when :

$$\frac{u}{l} = \frac{C^p n{-}1}{C^q n{-}1}$$

There are problems involving simultaneous equations of the more general type given by Mahāvīra, Āryabhaṭa II and Bhāskara II, Devarāja and others as follows :

$$by_1 = a_1x \pm c_1$$
$$by_2 = a_2x \pm c_2$$
$$by_3 = a_3x \pm c_3$$
$$\cdots\cdots\cdots$$

These problems are known as *saṃśliṣṭa kuṭṭaka* (the constant pulveriser). If α_1 and α_2 be the least values of x of the first and second equation and $b_1t_1 + \alpha_1$ and $b_2t_2 + \alpha_2$ are the corresponding general values of x satisfying the first and second equations respectively,

then $b_1t_1 + \alpha_1 = b_2t_2 + \alpha_2$

Solving t_1 and t_2 by the method of pulveriser, the general values of $b_1t_1 + \alpha_1$ i.e., of x satisfying both the equations are obtained. Next this is equated to the general value of x satisfying the third equation and the process is repeated.

Illustration :

Five heaps of fruits together with two fruits were divided equally among 9 travellers; 6 heaps together with 4 fruits were divided among 8; 4 heaps together with 1 fruit were divided among 7. Find the number of fruits in each heap. (*Gaṇitasārasaṃgraha*, ch. 6, v. 129½).

Solution : To solve

$$9y_1 = 5x + 2 \;\ldots\;\ldots\; (1)$$
$$8y_2 = 6x + 4 \;\ldots\;\ldots\; (2)$$
$$7y_3 = 4x + 1 \;\ldots\;\ldots\; (3)$$

From equation (1), we get $x \equiv 5 \pmod 9$
or $x = 9u + 5$ ... ... (4). Subst. in equation (2), we get

$$8y_2 = 6(9u + 5) + 4$$

$$\text{or} \quad 4y_2 = 27u + 17$$

From this $u \equiv 1 \pmod 4$ or $u = 4v + 1$.

Subst. this value of u in equation (4),
we get $x = 9(4v + 1) + 5$

$$\text{or} \quad x = 36v + 14 \ldots\ldots (5)$$

Subst. (5) in equation (3), we get,

$$7y_3 = 4x + 1 = 4(36v + 14) + 1$$

$$\text{or} \quad 7y_3 = 144v + 57 \ldots\ldots (6)$$

From here $v \equiv -2 \pmod 7$

$$\text{or} \quad v = 7p - 2.$$

Hence from (3) and (6),

$$4x + 1 = 1008p - 231$$

$$\text{or} \quad x = 252p - 58.$$

Least value $= 252 - 58 = 194$.

The problem of indeterminate equations was persued later by scholars like Nārāyaṇa, Kṛṣṇa, Kamalakara and various others.[1] The *Karaṇapaddhati* and *Yuktibhāṣā* have also discussed the rule.

The *Karaṇapaddhati* furnishes interesting information on the calculation of approximations of *mahāhāra* : *mahāguṇa* (i. e. $b : a$) and has applied the both upward (below-top) and downward (top-below) techniques to calculate the successive approximations. The

1. Bag, A. K. "The Method of Integral solution of Indeterminate Equations of the type : $by = ax + c$ in Ancient and Medieval India", *Indian Journal of History of Science*, **12** No. 1, p. 1–16, 1977.

relevant verse of the *Karaṇapaddhati*, (ii, 5) runs as follows :

anyonyaṃ vibhajenmahāguṇahārau yāvadvibhakte'lpata
tāvallabdhaphalāni rūpamapi onnyasyedadho'dhaḥ kramāt
prakṣipyāntyamupāntimena guṇite svordhve tadantyaṃ tyajed
bhuyo'pyeṣa vidhirbhaved guṇahārau syātām tadordhvasthitau ||

English Translation :

"The *mahāguṇa* (a) and *hāra* (b) should be simultaneously divided till it becomes negligible (i. e. zero). The respective quotients and a number one are placed one below the other. Then multiply the last but one (*upānta*) by the number placed above it (*upānta ūrdha*) and the last is added to it, and then this last number is left. This is the process of obtaining the *guṇa* and *hāra* placed in two *urdha* positions."

This method upward (below-top) technique is similar to that of Bhāskara I with the difference that the mutual division is repeated upto the remainder zero, like that of our modern method.

For example,

$$\text{let } \frac{b}{a} = \frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{16}}$$

The rule gives a process of calculating the final convergent $\frac{p_n}{q_n}$. Here for $n = 3$, $\frac{p_3}{q_3} = \frac{b}{a} = \frac{355}{113}$. This has been obtained from the quotients 3, 7, 16 and 1 placed one below another following the process discussed before.

The *Karaṇapāddhati*, (ii 6) has given an alternative (top-below) process of this rule for calculating the successive approximations of circumference : diameter, i. e. $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ ... etc., in the following rule.

anyonyāhṛtabhājyahārakaphalaṃ sarvaṃ tvadho'dho
nyasedekatrādyaphalena hīnamaparatraikam dvayaścapari|
kurjād valyupasaṃhṛtiṃ hyuparitaḥ pūrvapraṇāśaṃ vinā
tyājyaṃ tatprathamordhvagaṃ hāraguṇāśśiṣṭāśca vā
svecchayā||

English Translation :

"The *hāra* and *bhājya* (*guṇa*) are simultaneously divided and the results are all placed systematically one below the other in one place (first place). Put the partial quotients without the first quotient in another place (second place) and place one over both the places. Perform the *vallī* operations from the top and leave the number of the first place which was not destroyed in the operation. The *hāra*, *guṇa* and the remaining results (are obtained) as desired.

This is undoubtedly a reverse (top-below) technique of calculating $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ ...etc., followed before. Here the *vallī* operation is performed starting from top like that of our modern method of calculation as follows :

1st place	*hāra*	2nd place	*guṇa*
1			
a_1	a_1	1	1
a_2	a_1a_2+1	a_2	a_2
a_3	$a_3\{a_1a_2+1\}+a_1$	a_3	a_2a_3+1
a_4	$a_4[a_3\{a_1a_2+1\}+a_1]$	a_4	$a_4\{a_2a_3+1\}+a_2$

Hence the successive approximations of circumference (*hāra*) to diameter (*guṇa*), i.e. $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$...etc. are :

$$\frac{p_1}{q_1} = \frac{a_1}{1}, \frac{p_2}{q_2} = \frac{a_1a_2+1}{a_2}, \frac{p_3}{q_3} = \frac{a_3\{a_1a_2+1\}+a_1}{a_2a_3+1}$$

and so on.

The author of *Yuktibhāṣā* (vide appendix of the edited text) has used the same technique to calculate $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$... .

This shows that Indian scholars had a more or less distinct idea about the application of continued fraction and used the tool $p_nq_{n-1} - q_np_{n-1} = \pm 1$ for the solution $by = ax \pm c$, according as n is even or odd.

The discussion shows that Āryabhaṭa I (496 A. D.) first obtained a general solution of the problem $by = ax \pm c$. But Āryabhaṭa I himself admitted that he was discussing the knowledge current in Kusumpura (founded in the 1st century B. C.). Further we find that the Indian interest began during the time of *Śulbasūtras*. The process thus has a long history behind it and did not appear all of a sudden in a fully developed form. The subject of *Kuṭṭaka* was considered among the Indian mathematicians as an important branch of their algebra.

Regarding Greek contribution, Nicomachus of Geresa[1] (1st century A. D.) gave an example on the problem of remainders involving linear indeterminate

1. Dickson, L. E. *History of the Theory of Numbers*, 2, p. 58, New York, 1934.

analysis. Diophantus[1] (3rd century A. D.) discussed the indeterminate equations of the second degree and higher degrees but did not deal equations of the first degree. Preliminary notions of Greek geometry, according to Kaye[2], which is responsible for the evolution of the rule is not found common among Indian works as it has been traced in Greek works.

The basis for Chinese contribution is placed on the following example found in the *Sun–Tzu Suan Ching.*[3] "There are certain things whose number is unknown. Repeatedly divided by 3, the remainder is 2; by 5 the remainder is 3; and by 7, the remainder is 2. What will be the numbers ?". No process of finding them has been indicated. The solution can be obtained by inspection[4]. The application of the equation : $by = ax \pm c$ was first found in a calendrical work, *Ta–Yenli* of I–Hsing[5] (683–727 A. D.). The problems of indeterminate analysis found in *Ta-Yenli* was solved by I-Hsing with a method *Ta-Yen-Shu*[6] which was similar to the Indian method of *kuṭṭaka.*[7] I–Hsing

1. Heath, T. L. *A History of Greek Mathematics*, Diophantus of Alexandria, 2, p. 440–517, Oxford, 1921.
2. Kaye, G. R. "Notes on Indian Math.– No. 2 : Āryabhaṭa, *JASB*, 4, p 111–47, 1908.
3. Mikami, Y. *The development of Mathematics in China and Japan,* 1913; Needham, J. *Science and Civilisation in China*, p. 119–22, Cambridge, 1959.
4. Mazumdar, N. K. "On Chinese Indeterminate Analysis", *Bulletin of the Calcutta Mathematical Society* 5, p.9-11,1913-14.
5. Sarton, G. *Introduction to History of Science*, 1, "I-Hsing" p. 514.
6. Sarton, G. *Ibid.*, 1, p. 514.
7. Needham, J. *Ibid.*, 3, p. 119–22, Cambridge, 1959.

came to India in 673 A. D., became a Tantric-Buddhist monk and learnt Sanskrit. Hence it is quite possible that I–Hsing acquired the technique of solving indeterminate problems from Indian scholars and it is through his effort, the knowledge was carried to China.

Indeterminate Equations of Second Degree (Vargaprakṛti)

The term *vargaprakṛti* has been used by Indian scholars to designate the equations of the type $Nx^2 \pm c = y^2$. The most fundamental equation of this type has been regarded as $Nx^2 \pm 1 = y^2$. Brahmagupta (628 A. D.) first gave solutions in rational integers of both of these types. Further refinements, clarifications, extensions were made by subsequent mathematicians such as Śrīpati, Bhāskara II, *Nārāyaṇa* and others including several commentators who rendered considerable service to this branch of algebra. Here N is termed as *guṇakaprakṛti*, x : *kaniṣṭhapada*, *hrasvamūla*, or *ādyamūla*, y : *jyeṣṭhapada*, or *anyamūla*; c : *kṣepa*, *prakṣepa* or *prakṣepaka*.

The equation $Nx^2 + 1 = y^2$ is known by mistake now a days as Pell's equation (c. 1668 A. D.). The equation must be known as Brahmagupta–Bhāskara equation, for Bhāskara II gave a satisfactory solution of the nearly solved method of Brahmagupta. Lagrange gave a complete solution of the equation in 1767 A. D. with the help of continued fraction.

(A) *Solution of* $Nx^2 + 1 = y^2$ *(Brahmagupta)*–Find a set of integral values α and β of x and y and form an auxiliary equation of the form :

$$N\alpha^2 \pm k = \beta^2 \text{ for } k = \pm 1, \pm 2 \text{ or } \pm 4.$$

From these an unlimited number of integral so'utions can be readily obtained by principle of composition (*samāsabhāvanā*) discovered by Brahmagupta (*Brāhmasphuṭasiddhānta*, ch. 18, p. 64–55) and applied by later mathematicians.[1] In algebraic symbo's the principle of *samāsabhāvanā* may be expressed as fol'ows : For conveniently chosen values of c_1 and c_2, if (a_1, b_1) and (a_2, b_2) be a set of solutions of $Nx^2+c_1 = y^2$ and $Nx^2+c_2=y^2$ respectively, then $x=a_1b_2 \pm a_2b_1$ and $y=b_1b_2 \pm Na_1a_2$ wi l satisfy the equation $Nx^2+c_1c_2=y^2$. The aim of the scholars like Brahmagupta and Śrīpati was to obtain solutions of $Nx^2+1=y^2$ in positive integers. They however discovered that an integral solution of $Nx^2+k=y^2$ can always be found if $k=\pm 1, \pm 2$ or ± 4. Having got the solution, an infinite number of solution can be obtained by repeated application of *samāsabhāvanā*. This is undoubtedly a remarkable feat when we realise that this was done in 628 A. D. The method would have been perfect if only a method other than trial be avilable to Brahmagupta for the solution. Bhāskara I1 did it by method known as *cakravāla* as following.

(B) *Bhāskara II's Cakravāla Process :*

i. e. solution of $Nx^2+1=y^2$

hrasvajyeṣṭhapadakṣepānbhājyaprakṣepa bhājakān ||
kṛtvā kalpo guṇastatra tathā prakṛtitaścute |
guṇavarge prakṛtyone 'thavālpaṃ śeṣakam yathā ||
tattu kṣepahṛtam kṣepo vyastaḥ prakṛtitaścute |

1. Bhāṣkara II's *Bijagaṇita*, v. 71; *Bijagaṇitavataṃśa* of Nārayaṇa ed. K. S. Shukla, p. 36–44.

guṇalabdhiḥ padaṃ hrasvaṃ tato jyeṣṭhamato 'sakṛt ||
tyaktā pūrvapadakṣepāṅścakravālamidaṃ jaguḥ |
caturdvaikayutā vevamabhinne bhavataḥ pada ||
caturdvikṣepamūlābhyāṃ rūpakṣepārthabhāvanā |[1]

(*Bījagaṇita*, v. 75)

English Translation :

"Take the lesser root, greater root and interpolator of a square nature) as divident, interpolator and divisor. Select an arbitrary number so that its square diminished by *prakṛti* or the *prakṛti* diminished by square of the chosen number, (as the case may be) is the least. This (difference) divided by the original interpolator is the (new) interpolator; it should be reversed in sign in case of the subtraction from *prakṛti*. The quotient corresponding to that value of the multiplier is the (new) lesser root; likewise is obtained the greater root. The same process should be followed repeatedly putting aside (each time) the previous roots and the interpolator. This process is called *cakravāla* (or the cyclic method). By this method there will appear two integral roots corresponding to an equation with ± 1, ± 2 or ± 4 as interpolator. In order to derive integral roots corresponding to an equation with the additive unity from those of the equation with the interpolator ± 2 or ± 4, the principle of composition (should be applied)."

Select $Na^2 + k = b^2$, for any suitable k,

and $N.\ 1^2 + (m^2 - N) = m^2$

1. Bhāskara II's *Bījagaṇita*—Vargaprakṛti, v. 2–4.

then by *bhāvanā*, Bhāskara II obtained

$$N\left(\frac{am+b}{k}\right)^2+\frac{m^2-N}{k}=\left(\frac{bm+Na}{k}\right)^2 \ldots\ldots(1)$$

Next to choose the value of the m so that m^2-N is numerically smaller and $\frac{am+b}{k}$ becomes integer, since its value is determined by means of *kuṭṭaka*.

$$\text{Let } \frac{am+b}{k}=a_1,\ \frac{m^2-N}{k}=k_1$$

$$\frac{bm+Na}{k}=b_1$$

Proposition I:

When a_1 is an integer, b_1, k_1 are also integers. Then the equation (i) becomes,

$$Na^2_1+k_1=b^2_1$$

A new equation of the same kind, viz.

$Na^2_2+k_2=b^2_2$ can be obtained, proceeding in a similar way.

Proposition II:

After an finite number of repetitions, the equations.

$N\alpha^2+t=\beta^2$, when $t=\pm 1$, ± 2 or ± 4.

is available.

In order to obtain an integral solution of the original equation $Nx^2+t=y^2$ from those of the interpolators with ± 2, ± 4, the principle of composition is applied. The truth of these two propositions was perhaps understood by Bhāskara II in case of concrete instances, for no proof was given by him.

Examples :

(i) $61x^2 + 1 = y^2$
(ii) $67x^2 + 1 = y^2$ [*Bījagaṇita*, v. 76]

Solution :

(i) $61x^2 + 1 = y^2$
Select an auxiliary equation
$61.1^2 + 3 = 8^2$ ... ...(1)
Let $a_1 = 1$, $b_1 = 8$, $k_1 = 3$ and $N = 61$
Then by Bhāskara II's lemma we get

$$N\left(\frac{a_1m+b_1}{k_1}\right)^2 + \frac{m^2-N}{k_1} = \left(\frac{b_1m+Na_1}{k_1}\right)$$

where m is an arbitrary chosen number§. Here m is so chosen that $m^2 - N$ is least and $\frac{a_1m+b_1}{k_1}$ is an integer

Taking $m = 7$, we get $\frac{a_1m+b_1}{k_1} = 5$, $\frac{m^2-N}{k_1} = -4$.

and $\frac{b_1m+Na_1}{k_1} = 39$

Let $a_2 = 5$, $b_2 = 39$, $k_2 = -4$
Hence from (1), we get
$61.5^2 - 4 = 39^2$ ... ... (2)
Thus in one step, we get an equation
$Nx^2 + t = y^2$ when $t = -4$
From (2), $61.\left(\frac{5}{2}\right)^2 - 1 = \left(\frac{39}{2}\right)^2$

Apply the principle of *bhāvanā*,

$\frac{5}{2}$ $\frac{39}{2}$ -1

and $\frac{5}{2}$ $\frac{39}{2}$ -1

We get $\frac{195}{2}$, $\frac{1523}{2}$, 1............(3)

Again applying *bhāvanā* between

$\frac{195}{2}$ $\frac{1523}{2}$, 1

and $\frac{5}{2}$ $\frac{39}{2}$ -1

We get 3805, 29718, -1 ....(4)

Finally applying *bhāvanā* in this with itself,

i.e. 3805 29718, -1

3805 29718 -1

We get 226153980, 1766319049, 1

Hence $x = 226153980$

$y = 1766319049$

is the solution of $61x^2 + 1 = y^2$

Bhāskara II gave no proofs of Propositions I and II. Proofs are given by Datta and Singh[1], Hankel[2] and Krishnaswami Ayangar[3].

(ii) Solution of : $67x^2 + 1 = y^2$

We know, $67.1^2 - 3 = 8^2$... ... ...(1)

Comparing with $Na^2 + c = b^2$, $N = 67$ $a = 1$, $b = 8$, $c = -3$.

Now $\frac{am+b}{c} = \frac{1.m+8}{-3} = \frac{m+8}{-3}$ is an integer when $m = 7$.

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1. Datta, B and Singh, A. *Ibid.*, 2., p. 157–166.
2. Hankel, H. *Zur Geschichte der Mathematik*, p. 202, Leipzig, 1874.
3. Krishṇaswāmī Ayyangar, A. A. *Journal of the Indian Mathematical Society*, 18, first series, second part, p. 232 45.

Hence, taking $m = 7$

$$a_1 = \frac{am+b}{c} = \frac{7+8}{-3} = -5$$

$$b_1 = \frac{bm+Na}{c} = \frac{8.7+67}{-3} = \frac{56+67}{-3} = \frac{123}{-3} = -41$$

$$c_1 = \frac{m^2-N}{c} = \frac{49-67}{-3} = \frac{-18}{-3} = 6$$

$$\therefore 67.(-5)^2+6 = (-41)^8$$

$$\text{or } 67.5^2+6 = 41^2. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence we get,

$$67.5^2+6 = 41^2$$

Now, $a_2 = \dfrac{5p+41}{6}$ is an integer for $p = 5$

$$\therefore a_2 = \frac{66}{6} = 11$$

$$b_2 = \frac{41.(+5)+67.5}{6} = \frac{+205+335}{6} = \frac{540}{6} = 90$$

$$c_2 = \frac{25-67}{6} = \frac{-42}{6} = -7$$

Hence, we get $67.11^2-7 = 90^2$.................(3)

Now, $a_3 = \dfrac{11t+90}{-7}$ is an integer for $t = 9$

$$\therefore a_3 = \frac{99+90}{-7} = \frac{189}{-7} = -27$$

$$b_3 = \frac{90t+67.11}{-7} = \frac{90\times 9+737}{-7} = \frac{1547}{-7}$$

$$= -221$$

$$c_3 = \frac{81-67}{-7} = \frac{14}{-7} = -2$$

We get, $67.(-27)^2-2 = (-221)^2$

or $67(.27)^2-2 = (221)^2$ ... ...(4)

Applying the *samāsa* between :

27	221	-2
27	221	-2

We get $X = 27 \times 221 + 27 \times 221 = 11934$,
$Y = 221 \times 221 + 67 \times 27 \times 27 = 97684$, satisfying the equation, $67.X^2 + 4 = Y^2$

or $67.(11934)^2 + 4 = (97684)^2$ ... ...(5)

Dividing both sides by 4,

or $67.(5967)^2 + 1^2 = (48842)^2$... ... ...(6)

$\therefore$ $x = 5967$ and $y = 48842$

From modern approach we know :

$$\sqrt{67} = 8 + \frac{1}{5+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{7+}\ \frac{1}{8+}\ldots\ldots\ldots$$

$$= 8,\ \frac{41}{5},\ \frac{90}{11},\ \frac{221}{27},\ \frac{97684}{11939}\left(=\frac{48842}{5967}\right),\ \ldots\ldots\ldots$$

For the solution of $67x^2 + 1 = y^2$, if 1, m be the solution then according to Bhāskara II's method $k = m^2 - N$ i.e. if (1, 8) be an an arbitrary solution, then $k = 8^2 - 67 = -3$ Hence the first set can be written as :

1) $67.1^2 - 3 = 8^2$. Likewise the other sets are
2) $67.5^2 + 6 = (41)^2$
3) $67.(11)^2 - 7 = (90)^2$
4) $67.(27)^2 - 2 = (221)^2$
5) $67.(11934)^2 + 4 = (97684)^2$
and 6) $67.(5967)^2 + 1 = (48842)^2$

There values for x and y given by Bhāskara II are the successive convergents of $\sqrt{67}$.

Problems of the following type with results were given by Brahmagupta, Bhāskara II, Nārāyaṇa, Devarāja, a few of which are given here.

(1) $92x^2+1=y^2$ (Brahmagupta)
(2) $83x^2+1=y^2$ (Brahmagupta)
(3) $11x^2+1=y^2$ (Bhāskara II)
(4) $61x^2+1=y^2$ (Bhāskara II)
(5) $103x^2+1=y^2$ (Nārāyaṇa)
(6) $97x^2+1=y^2$ (Nārāyaṇa)

(C) *Rational solution* of $Nx^2+1=y^2$

(i) By *Brahmagupta (Brāhmasphuṭasiddhānta*, ch. 18, v. 65. *)*

For conveniently chosen values of K and K^1, if (a, b) and (a^1, b^1) be a set of solutions of $Nx^2+K=y^2$ ......(1) and $Nx^2+K^1=Y^2$......(2) then $N(ab^1 \pm a^1b)^2+KK^1=(bb^1 \pm Naa^1)^2$.

It, therefore, follows that :

$$N\left(\frac{ab^1 \pm a^1b}{\sqrt{KK^1}}\right)^2 + 1 = \left(\frac{bb^1 \pm Naa^1}{\sqrt{KK^1}}\right)^2$$

If KK^1 be a perfect square,

$$x=\frac{ab^1 \pm a^1b}{\sqrt{KK^1}}, \quad y= \pm \frac{bb^1 \pm Naa^1}{\sqrt{KK^1}}$$ are rational

solutions of $Nx^2+1=y^2$. If KK^1 be not a perfect square, the relation (2) is to be replaced by relation (1). In particular, if we put $K=K^1$, the roots become $\frac{ab \pm ab}{K}$ and $\pm\frac{b^2 \pm Na^2}{K}$. The lower sign gives the root $(0, \pm 1)$ which must be left out of consideration.

(ii) *By an unknown mathematician* :

An anonymous scholar gave the rational solution of the equation $Nx^2+1=y^2$ in the form :

$$x=\frac{k}{Bh-Ab}, \quad y=\frac{Ah-Bb}{Bh-Ab}$$

where b, k, and h are respectively the base, upright and hypotenuse of a rightangled triangle; A, B being two numbers such that $A^2 - B^2 = N$.

The solution is unique and *is the most* general rational solution. The solutions given by other mathematicians namely Brahmagupta (628 A. D.), Śrīpati (1039 A. D.), Bhāskara II (1150 A. D.) Nārāyaṇa (1357 A. D.), Jnānarāja (1503 A. D.), Kamalākara (1658 A. D.), and also those given by John Wallis and W. Browncker (1657 A. D.) are easily deducible from it[1].

(D) *Solution of* $NX^2 + c = Y^2$ (*Brāhmasphuṭasiddhānta*, ch. 18, v. 66).

If (p, q) be any rational solutions and if (r, s) be a solution of $NX^2 + 1 = Y^2$, then by the principal of *samāsa* : $X = ps \pm qr$, $Y = qs \pm Npr$ is a solution of the given equation. The process when repeated gives an infinite number of solution.

(E) *Solution of* $Mn^2x^2 + c = Y^2$ (*Brāhmasphuṭasiddhānta* ch. 18, v. 70).

The equation is transformed to $Mu^2 \pm c = Y^2$ when $nx = u$. Hence if (u, y) be a rational solution of the latter, $\left(\frac{u}{n}, y\right)$ is a rational solution of the former.

(F) *Solution of* $a^2x^2 \pm c = Y^2$ (*Brāhmasphuṭasiddhānta*, ch. 18, v. 69).

Now, $(Y - ax)(Y + ax) = \pm c$. Let $Y - ax = m$, then $Y + ax = \pm \frac{c}{m}$. Hence, $x = \frac{1}{2a}\left(\pm \frac{c}{m} - m\right)$,

1. Shukla, K. S. *Gaṇita*, 1, No. 2, p. 1-12; compare also with Shukla's edition of *Pāṭīgaṇita*, p. 159, lines 9-12.

$Y = \frac{1}{2}\left(\pm \frac{c}{m} + m\right)$ are rational solutions where m is any rational number.

Bhāskara II and later mathematicians gave solutions of more general equations of the second degree. Bhāskara II exhibited a remarkable ingenuity by reducing them to the form $Nx^2 + 1 = y^2$. The methods which though have been indicated mostly to arrive at rational solutions, also hint to get the integral solutions in special problems. The following general types are noteworthy :–

(1) $ax^2 + bx + c = y^2$ (Colebrooke, *Algebra*, p. 245).

Hints : This transforms to $\left(ax + \frac{b}{2}\right)^2 = ay^2 + \frac{1}{4}(b^2 - 4ac)$ (after Bhāskara II)

Putting $Z = ax + \frac{b}{2}$, $t = \frac{1}{4}(b^2 - 4ac)$, we get $ay^2 + t = Z^2$. This is the well known *vargaprakṛti* eqn. The solution can be obtained by the previous method.

(2) $ax^2 + bx + c = a^1y^2 + b^1y + c^1$ (Colebrooke, *Algebra*, p. 250).

Hints : Completing the square on one side, say on the left, the equation reduces to, $(ax + \frac{1}{2}b)^2 = aa^1y^2 + ab^1y^2 + ab^1y + (ac^1 + \frac{1}{4}b^2 - ac)$ or $aa^1y^2 + ab^1y^2 + k = z^2$.

where $k = ac^1 + \frac{1}{4}b^2 - ac$ and $z = ax + \frac{1}{2}b$.

This reduced equation can obviously be solved by the method as described in (1).

(3) $ax^2 + by^2 + c = z^2$ (Colebrooke, *Algebra*, p. 250).

Bhāskara II indicated several hints for solving the equation.

(i) Let $x = my$, the equation becomes :

$$z^2 = (am^2 + b)y^2 + c = ky^2 + c \text{ where } k = am^2 + b.$$

This is well-known *vargaprakṛti* type.

(ii) Let $x = my + n$, the equation reduces to type (1) i.e., $z^2 = ky^2 + 2amny + t$

where $k = am^2 + b$ and $t = an^2 + c$.

(iii) Let $by^2 + c = W^2$, hence $Z^2 - W^a = ax^2$

Putting $Z - W = mx$, $Z + W = \frac{a}{m}x$

we get $Z = \frac{1}{2}\left(m + \frac{a}{m}\right)x$, $W = \frac{1}{2}\left(\frac{a}{m} - m\right)x$,

From any solution of $by^2 + c = W^2$, we can thus obtain a solution for x and y of the given equation.

(iv) When $c = 0$, the equation becomes $ax^2 + by^2 = z^2$ Putting $x = uy$, $z = vy$, the equation transforms to $au^2 + b = v^2$.

Thus in every way the solution of the equation depends on the solution of the *vargaprakṛti*.

(4) $ax^2 + bxy + cy^2 = Z^2$ (Colebrooke, *Algebra*, p. 251–52).

Case (i) Let $a = p^2$

$$\text{then } \left(px + \frac{by}{2p}\right)^2 = Z^2 - y^2\left(c - \frac{b^2}{4p^2}\right)$$

$$\text{or } Z^2 - W^2 = y^2\left(c - \frac{b^2}{4p^2}\right) \text{ where } W = px + \frac{by}{2p}$$

Now, putting $Z - W = my$, $Z + W = \left(c - \frac{b^2}{4p^2}\right)\frac{y}{m}$, we get Z and W, consequently, we get x, y and Z.

The same method holds if c be a perfect square.

Case (ii) Neither a nor c is a perfect square.

Multiplying the given equation by a, we get,

$$(ax+\tfrac{1}{2}by)^2 = aZ^2 - ky^2$$

or $W^2 = az^2 - ky^2$ where $W = ax+\tfrac{1}{2}by$.

The equation is of the type (3) when $c = 0$.

The solution of the following examples were also given by the Indian scholars :–

(1) $3x^2+6x=y^2+2y$ (*Ibid., Algebra*, p. 251).

(2) $7x^2+8y^2=Z$ (Colebrooke, *Algebra*, p. 252).

(3) $7x^2-8y^2+1=+Z$ (Colebrooke, *Algebra*, p. 252).

(4) $6x^2+2x=Y^2$ (Colebrooke, *Algebra*, p. 247).

(5) $3\sqrt{\frac{xy+y}{2}}+\sqrt{x^2+y^2}+\sqrt{x^2-y^2+8}+\sqrt{x+y+2}$

$+\sqrt{x-y+2}=Z^2$. (Colebrooke, *Algebra* p. 255).

Hints : putting for (i) $x=u^2-1$, $y=2u$, (ii) $x=v^2+2v$, $y=2v+2$, (iii) $x=v^2-2v$, $y=2v-2$ or (iv) $x=v^2+4v+3$, $y=2v+4$ separately in the equation, the reduced equation obtained will always depend upon the solution of the Pellian equation.

Bhāskara II, Nārāyaṇa and other later mathematicians gave problems involving equations of the higher degrees Some of them are simultaneous equations, but the solution of all of them has been tried by reducing them to the *vargaprakṛti* with suitable devices.

CHAPTER V

TRIGONOMETRY

The work of Hipparchus (c. 150 B. C.), Menelaus of Alexandria (c. 100 A. D.) and Ptolemy (c. 150 A. D.) in connection with astronomical problems in the early Christian Era led to the growth of trigonometrical knowledge in Greece in a rather fragmentary character. It was however at a later date (4th to 12th century A.D.) the Indian developed the subject in a systematic manner almost resembling its modern form. This was then transmitted to the Arabs by the beginning of 9th century A.D., who introduced some further improvement. From the Arabs, the knowledge went to Europe where a detailed account of trigonometrical knowledge first appeared in the work *De Triangulis* of Regiomontanus (1533 A.D.).

In ancient Indian Mathematics, Trigonometry forms an integral part of Astronomy. It is undoubtedly a later development. Earliest reference to trigonometrical concept is found in the *Sūryasidhānta* (c. 400 A. D.), *Pañcasiddhāntikā* of Varāhamihira (c. 505 A.D.) *Brāhmasphuṭasiddhānta* of Brahmagupta (c. 628 A.D.) *Śiṣyadhivṛddhida* of Lalla (768 A.D.). More or less systematic study on the subject was made by Bhāskara II (c. 1150 A. D.) in the section on *Jyotpatti* in his *Siddhāntaśiromaṇi* and by Kamalākara in the *Sphuṭādhyāya* chapter of his *Siddhāntatattvavīveka.*

Bhāskara II[1] laid particular emphasis on the study of trigonometry in order to obtain proficiency in astronomy. The idea on values of π, sine, cosine expressed in series was first found in the works of Mādhava and then appeared in the *Karaṇa Paddhati*, *Tantrasaṃgraha* and *Yuktibhāṣā*

Preliminary Indian Methods in Plane Trigonometry

(1) *Trigonometical functions* :—In India, like Babylon and Greece, the study of the properties of the right-angled triangle inscribed in a circle in connection with the use of gnomon in astronomical measurement was given much importance The circle was divided into quadrants and circumference into many equal parts. Three functions[2] *Jyā*, *Kojyā* and *Utkramajyā* were used in the development of this new geometry, known as plane trigonometry. The modern equivalents of these functions (Fig. 18) are as follows :—

Jyā A = PM = R Sin A

Kojyā A = OM = R Cos A

Utkramajyā A = MB = OB − OM

= R − R Cos A

= R(1 − Cos A)

= R Versin A

where R is the Radius[3] of the circle.

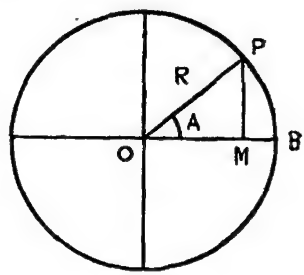

Fig. 18

1. *ācāryāṇām padavīṃ jyotpattyā jñātayā yato yāti*

(*Siddhāntaśiromaṇi-jyotpatti*, sl. 1)

"The *ācārya* (master) title (in astronomy) is offered on him who has acquired sufficient knowledge of trigonometry".

2. Synonymous terms of *Jyā* are *bhujajyā*, *jīvā*, *maurvi*, *kramajyā*, *dorjyā* etc., and of *Kojyā* are *koṭijyā*, *koṭi*. By definition, it denotes the compliment of the sine arc.

3. *śiñjini*, *trijyā* etc. (synonymous terms).

(2) *Calculation of R Sin* A, $A \leqslant 90^{\circ}$

Varāhamihira (c. 505 A. D.) in his *Pañcasiddhāntikā*[1] gave the following formulae :

$$R \text{ Sin } 30^{\circ} = \frac{R}{2}$$

$$R \text{ Sin } 60^{\circ} = \frac{\sqrt{3}}{2} R$$

$$R \text{ Sin } 90^{\circ} = R = 120'$$

$$(R \text{ Sin } A)^2 = \frac{R}{2} (R - R \text{ Cos } 2A)^2$$

and $(R \text{ Sin } A)^2 + (R \text{ Cos } A)^2 = R^2$

With the help of these formulae Varāhamihira perhaps calculated the values of R Sines for arcual measure ranging from 3° 45′ to its 24 multiples i.e ($3^{\circ}45' \times 24 = 90^{\circ}$).

Bhāskara I (c 600 A. D.) in his *Mahābhāskarīya* (ch. 7, v. 17–19) has also set the following approx-mimate formula for calculating the R Sine of an acute angle without the use of a table.

$$R \text{ Sin } A = \frac{R(180^{\circ} - A) A}{[40500 - (180^{\circ} - A)A] \frac{R}{4}}$$

where A is in degrees. Equivalent forms of the formula have been given by almost all subsequent scholars of mathematics and astronomy. The formula in modern notation may be written as

$$\text{Sin } B = \frac{16B (\pi - B)}{5\pi^2 - 4B(\pi - B)} \quad \text{where B radians}$$

1. ch. 4. v. 19.

correspond to A degrees.

Putting $B = \frac{\pi}{3}$, and $\frac{\pi}{7}$ we get

$\text{Sin } \frac{\pi}{3} = .8643...$, $\text{Sin}\left(\frac{\pi}{4}\right) = .70058...$,

and $\text{Sin}\left(\frac{\pi}{7}\right) = .4313...$ which are correct upto 2 places of decimals. The values of $\text{Sin } \pi$, $\text{Sin } \frac{\pi}{2}$ and $\text{Sin } \frac{\pi}{3}$ come out to be accurate. Bhāskara I ascribes this formula to Āıyabhaṭa I[1]. It occurs in the *Brāhmasphuṭasiddhānta* and in several later works also.[2]

(3) *Calculation of Sin A*, $A < 90^\circ$

Bhāskara I (c. 600 A D.), Brahmagupta and others have made use of the formulae :

$$R \text{ Sin } (90^\circ + A) = R \text{ Sin } (90^\circ - A) = R \text{ Cos } A$$

$$R \text{ Sin } (180^\circ + A) = -R \text{ Sin } A$$

$$R \text{ Sin } (270^\circ + A) = -R \text{ Sin } (90^\circ - A) = -R \text{ Cos } A$$

where $A \leqslant 90^\circ$

The basis of the above formulae may be explained as follows :

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1. See his commentary on *Āryabhaṭīya*, i. 11.
2. Gupta, R. C. 'Bhāskara I's Approximation to Sine' *Indian Journal of History of Science*, 2, No. 1, p. 121–36, 1967.

Let AOB, BOC, COD and DOA be four quadrants of a mean circular orbit of a planet, A being the Planet's apogee from which the mean anomaly is measured anticlockwise. If P_1, P_2, P_3 and P_4, be the Planet's positions at four quadrants respectively, then

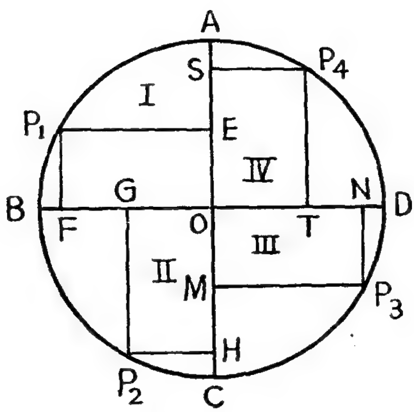

Fig. 19

In *Quadrant I*

arc traversed = A P_1 = *bhūja*; arc to be traversed = P_1 B = *koṭi*

In Quadrant II

arc traversed = BP_2 = *koṭi*; arc to be traversed = P_2C = *bhūja*

In Quadrant III

arc traversed = CP_3 = *bhūja*; arc to be traversed = P_3D = *koṭi*

In Quadrant IV

arc traversed = DP_4 = *koṭi*; arc to be traversed = P_4A = *bhūja*

The R Sine of the *bhūja* is defined to be the R Sine of the corresponding anomaly and the R Sine of the *koṭi* is defined to be the R Cosine of the corresponding anomaly. The R Sine is positive in the first and second quadrants and negative in the third and fourth quadrants. The R cosine is positive in the first and fourth quadrants and negative in the second and third quadrants. Then we have,

$$R \text{ Sin } \widehat{AP_1} = P_1E \qquad R \text{ Cos } \widehat{AP_1} = P_1F$$

$$R \text{ Sin } \widehat{AP_2} = P_2H \qquad R \text{ Cos } \widehat{AP_2} = -P_2G$$

$$R \text{ Sin } \widehat{AP_3} = -P_3M \qquad R \text{ Cos } \widehat{AP_3} = -P_3N$$

$$R \text{ Sin } \widehat{AP_4} = -P_4S \qquad R \text{ Cos } \widehat{AP_4} = PT_4$$

This is equivalent to

$R \text{ Sin } (90^\circ + A) = R \text{ Sin } (90^\circ - A) = R \text{ Cos } A$

$R \text{ Sin } (180^\circ + A) = -R \text{ Sin } A$

$R \text{ Sin } (270^\circ + A) = -R \text{ Sin } (90^\circ - A) = -R \text{ Cos } A$

Where $A < 90^\circ$

Similarly R Cosine formulae can also be deduced.

Bhāskara I[1] has given several other formulae which may be given without proof as follows :

$R \text{ Sin } (90^\circ + A) = R \text{ Sin } 90^\circ - R \text{ versin } A$

$R \text{ Sin } (180^\circ + A) = 90^\circ - R \text{ versin } 90^\circ - R \text{ Sin } A$

$R \text{ Sin } (270^\circ + A) = R \text{ Sin } 90^\circ - R \text{ versin } 90^\circ$

$- R \text{ Sin } 90^\circ + R \text{ versin } A$ where $A < 90^\circ$

These formulae, though not explicitly stated, were given earlier by Āryabhaṭa I in his *Āryabhaṭīya* (iii, 22, 1st part see commentaries also) and used by Brahmagupta (*Brāhmasphuṭasiddhānta*, ii, 15–16).

Deduction of Formulae

In Varāḥmihira's work, we find the following formulae :

I. 'The *Jyā* of an arc diminished from a quadrant is the square-root of the difference of the square of the radius and the square of the *Jyā* of the arc',[2]

1. *Mahābhāskarīya*, iv. 2.

2. *Pañcasiddhāntikā*, ch. 4. v. 23 (1st part of the rule)

That is,

$$Jy\bar{a}\left(\frac{\pi}{2}-A\right)=\sqrt{R^2-Jy\bar{a}^2A}$$

$$\text{or, } R\,\text{Sin}\left(\frac{\pi}{2}-A\right)=\sqrt{R^2-(R\,\text{Sin}\,A)^2}$$

Expressed in modern form, it becomes,

$$\text{Sin}\left(\frac{\pi}{2}-A\right)=\sqrt{1-\text{Sin}^2A}$$

The rule gives the correct result.

II. 'Twice any desired arc is substracted from 90°; the *Jyā* of the remainder is subtracted from the radius. The square-root of the result multiplied by sixty (i.e. half, of the radius) is the *Jya* of that arc. By deducting that square (i.e. $Jy\bar{a}^2A$) from the square of the radius, the square of *Kojyā* is obtained.[1]

That is,

$$(Jy\bar{a}\ A)^2=\frac{R}{2}\left[R-Jy\bar{a}\left(\frac{\pi}{2}-2A\right)\right]$$

$$\text{or, } (Jy\bar{a}\ A)^2=\frac{R}{2}\left[R-Kojy\bar{a}\ 2A\right]$$

$$\text{and } R^2-(Jy\bar{a}\ A)^2=(Kojy\bar{a}\ A)^2$$

$$\text{or, } Jy\bar{a}^2\ A+Kojy\bar{a}^2\ A=R^2$$

These two rules are equivalent respectively to our modern formulae :

$$\text{Sin}^2A=\tfrac{1}{2}(1-\text{Cos}\ 2A) \text{ and } \text{Sin}^2A+\text{Cos}^2A=1$$

III. 'To find the *Jyā* of any desired arc, subtract from the radius, the *Jyā* of quarter of a circle reduced by double the arc and then add the square of half the

1. *ichāṃśadviguṇona tribhojyayonā trayusya cāpajyā* ।
ṣaṣṭiguṇā sā karaṇi tayā dhruvonāvaśeṣasya ॥
(*Pañcasiddhāntikā*, ch. 4, v. 5)

result to the square of half the *Jyā* of the double arc followed by the square root of this summation result'.[1]

This gives,

$$(Jy\bar{a}\ A)^2 = \left[\frac{Jy\bar{a}\ 2A}{2}\right]^2 + \left[\frac{R - Jy\bar{a}\left(\frac{\pi}{2} - 2A\right)}{2}\right]^2$$

This corresponds to our modern formula,

$$\text{Sin}^2 A = \frac{\text{Sin}^2\ 2A}{4} + \frac{\text{Versin}^2\ 2A}{4}$$

Lalla (c. 768 A. D.) gave the following rules about the relationship of trigonometrical functions :

1. 'The square of the *Jyā* is subtracted from the square of the radius, the square root of the remainder is the *Kojyā*, or *Kojyā* is the *Jyā* of a quadrant minus *bhūja* arc'[2]

Which means,

i) $\sqrt{R^2 - (Jy\bar{a}\ A)^2} = Kojy\bar{a}\ A$

and ii) $Kojy\bar{a}\ A = Jy\bar{a}\left(\frac{\pi}{2} - A\right)$

These are identical with modern formulae respectively as follows :

$$\sqrt{1 - \text{Sin}^2 A} = \text{Cos } A \text{ and } \text{Cos } A = \text{Sin}\left(\frac{\pi}{2} - A\right)$$

In Brahmagupta's *Brāhmasphuṭasiddhānta*, we find the following relations :

---

1. *śeṣeṣviṣṭeṣu dhanurdviguṇapadāyojya śeṣaguṇahīnā* ।
trijyā tadarddhāvargodviguṇajyārddhasya saṃyojyaḥ ॥
(*Pañcasiddhāntikā*, ch. 4, v. 3)

2. *Śiṣyadhivṛddhida*, ch. 2, v. 30.

1. Half the diameter diminished by the *Utkramajyā* of an arc or its complement is the *Jīva* or *Jyā* of the respective complementary arc. The square-root of the difference of the square of the radius and that of the *Jyā* of an arc or its complements is the *Jyā* of the respective complementary arc'.[1]

This can be written as :

i) $\textit{Jyā} \left(\frac{\pi}{2} - A\right) = R - \textit{Utkramajyā}\ A$

ii) $\textit{Jyā}\ A = R - \textit{Utkramajyā} \left(\frac{\pi}{2} - A\right)$

and iii) $\sqrt{R^2 - (\textit{Jyā}\ A)^2} = \textit{Jyā}\left(\frac{\pi}{2} - A\right)$

The modern equivalents are respectively,

$$\text{Sin} \left(\frac{\pi}{2} - A\right) = \text{Cos}\ A$$

$$\text{Sin}\ A = \text{Cos} \left(\frac{\pi}{2} - A\right)$$

and, $\sqrt{1 - \text{Sin}^2 A} = \text{Sin} \left(\frac{\pi}{2} - A\right)$

In the *Mahāsiddhānta* of Āryabhaṭta II, we find the following formulae :

1. 'The square-root of half the result obtained by adding to or subtracted from the maximum value of the *Jyā*, the product of the radius and the *Jyā* of an arc gives the *Jyā* of an arc equal to twelve times the

1. *vyāsadalamitara jīvābhujakoṭyaṃsotkramajyāyāhīnaṃ* ।
koṭibhujayāvyāsārddhakṛtiviśeṣātpadaṃcānyā ॥
(*Brahmasputasiddhānta*, ch. 14, v. 7)

first arc (i.e., $3^0 45' \times 12 = 45^0$) increased or decreased by half of the arc (under consideration)'.[1]

That is,

$$Jy\bar{a}\left(\frac{\pi}{4} \pm \frac{A}{2}\right) = \sqrt{\frac{R^2 \pm R\ Jy\bar{a}\ A}{2}}$$

Expressed in modern form, it becomes,

$$\text{Sin}\left(\frac{\pi}{4} \pm \frac{A}{2}\right) = \sqrt{\frac{1 \pm \text{Sin}\ A}{2}}$$

Bhāskara II[2] (1150 A. D) made a comprehensive list of the trigonometrical formulae (vide *Jyotpatti* section) including those of the previous workers, as given below. These formulae were found arranged serially and shows a genetic relation among them.

1) $R^2 - (R\ \text{Sin}\ A)^2 = (R\ \text{Cos}\ A)^2$ ...... (v.4).

2) $R\ \text{Sin}\ A = R\ \text{Cos}\left(\frac{\pi}{2} - A\right)$ ...... (v. 5).

3) 'Half the root of the sum of the squares of *Kramajyā* and *Utkramajyā* of any arc is the *Jyā* of half the arc; or, the square-root of half the product of the radius and the *Utkramajyā* is the *Jyā* of half that arc.'[3]

---

1. *iṣṭajyāgajyāhatihīnāḍhyauparamaśiñjinīvargo* ।
taddalamūle te staḥ saṃkhyārdhonāḍhyakarabhavau pindau ॥
(*Mahāsiddhānta*, ch. 3, v. 2)
gajya = trijyā, paramasiñjinī = trijyā, hināḍhyau = rahita sahitau, saṃkhyārdhonāḍhyakarabhavau = abhiṣṭardhena hinā yuktāśca dvādaśa śeṣasaṃkhyāsamau ($3^0 45' \times 12 = 45^0$).
2. Vide *Jyotpatti* section of *Siddhāntaśiromaṇi* of Bhāskara II.
3. *kramotkramajyākṛtiyogamulād dalam tadardhaṃsaka śiñjinī syāt* ।
trijyotkramajyā nihatirdalasya mulam tadardhaṃsaka śiñjinī vā ॥ (v. 10.)

These are :

$$\text{i) } \tfrac{1}{2}\sqrt{\textit{Jyā}^2 \text{ A} + \textit{Utkramajyā}^2 \text{ A}} = \textit{Jyā}\ \frac{\text{A}}{2}$$

$$\text{and ii) } \sqrt{\frac{\text{R} \times \textit{Utkramajyā} \text{ A}}{2}} = \textit{Jyā}\ \frac{\text{A}}{2}$$

In Modern notation, the formulae are :

$$\tfrac{1}{2}\sqrt{\text{Sin}^2 \text{ A} + \text{Versin}^2 \text{ A}} = \text{Sin}\ \frac{\text{A}}{2}$$

$$\text{and } \sqrt{\frac{\text{Versine A}}{2}} = \text{Sin}\ \frac{\text{A}}{2}$$

$$\text{4) } \textit{Jyā}\left(\frac{90^\circ \pm \text{A}}{2}\right) = \sqrt{\frac{\text{R}^2 \pm \text{R} \times \textit{Jyā} \text{ A}}{2}} \ldots \quad \ldots \text{(v.12)}$$

In modern notation, these are equivalent to,

$$\text{Sin}\left(\frac{90^\circ \pm \text{A}}{2}\right) = \sqrt{\frac{1 \pm \text{Sin A}}{2}}$$

5) 'Square of the difference of the *Jyās* of the two desired arcs is to be added with the square of the difference of their *Kojyās*, the square-root of this (result) when halved gives the *Jyā* of (an arc equal to) half the difference of those two arcs.'[1]

That is,

$$\tfrac{1}{2}\sqrt{(\textit{Jyā} \text{ A} - \textit{Jyā} \text{ B})^2 + (\textit{Kojyā} \text{ A} - \textit{Kojyā} \text{ B})^2} = \textit{Jyā}\ \frac{\text{A} - \text{B}}{2}$$

or, in modern notation, this comes out as,

1. *yaddorjayorantaramiṣṭayoryat kṛtijyāyontat kṛtiyoga mūlaṃ* ।
dalikṛtaṃ syād bhūjayorviyoga-khaṇḍasyajivaivamanekadhā vā ॥
(v. 13)

$$\tfrac{1}{2}\sqrt{(\text{Sin A} - \text{Sin B})^2 + (\text{Cos A} - \text{Cos B})^2} = \text{Sin}\ \frac{\text{A} - \text{B}}{2}$$

6) 'The square-root of half the square of the difference of *Jyā* and *Kojyā* of an arc is equal to the *Jyā* of another arc equal to half the difference of the arc and its complement'.[1]

That is,

$$\sqrt{\frac{(Jy\bar{a}\ \text{A} - Kojy\bar{a}\ \text{A})^2}{2}} = Jy\bar{a}\left[\frac{(90^\circ - \text{A}) - \text{A}}{2}\right]$$

It may be written in modern form as follows :

$$\sqrt{\frac{(\text{Sin A} - \text{Cos A})^2}{2}} = \text{Sin}\left[\frac{(90^\circ - \text{A}) - \text{A}}{2}\right]$$

7) $\text{R Sin}\ (90^\circ - 2\ \text{A}) = \text{R} - \frac{(\text{R Sin A})^2}{\text{R}/2}$ ...... (v.15)

8) '*Jyās* of two desired arcs are multiplied by the *Koṭijyās* of the other, each product is divided by the radius; the sum of the quotients is the *Jyā* of the sum of the angles and their difference is the *Jyā* of their difference.'[2]

That is,

$$Jy\bar{a}\ (\text{A} \pm \text{B}) = \frac{Jy\bar{a}\ \text{A} \times Kojy\bar{a}\ \text{B}}{\text{R}} \pm \frac{Jy\bar{a}\ \text{B} \times Kojy\bar{a}\ \text{A}}{\text{R}}$$

---

1. *doḥkoṭijīvāvivarasya vargo dalīkṛtastasya padena tulya* ।
syātkoṭibāhorvivarārdhajīvā vakṣyetha mūlagraṇam vināpi ॥
(v. 14).

2. *cāpayoriṣṭyordorjyemithaḥkoṭijyākahate* ।
trijyābhakte tayoraikyam taccāpaikasya dorjakāḥ ॥
(v. 21 and 22½).

In modern equivalents, these reduce to :

$$\text{Sin}\ (A \pm B) = \text{Sin}\ A\ \text{Cos}\ B \pm \text{Cos}\ A\ \text{Sin}\ B$$

These two relations were called respectively *samāsa bhāvanā* and *antara bhāvanā* by Bhāskara II (1150 A. D.). Almost the same list of trigonometrical formulae of Bhāskara II appears in the work *Siddhānta-tattvaviveka* of Kamalākara besides many other important formulae.[1] The *Karaṇapaddhati* (ch vi, vs. 9–11) gives also some correct formulae as follows :

i) $\sqrt{R^2 \pm R.\ R\ \text{Sin}\ A} = R\left(\text{Sin}\ \frac{A}{2} \pm \text{Cos}\ \frac{A}{2}\right)$

ii) $R\text{Sin}\ \frac{A}{2} = \frac{R\left(\text{Sin}\ \frac{A}{2} + \text{Cos}\frac{A}{2}\right) + R\left(\text{Sin}\ \frac{A}{2} - \text{Cos}\frac{A}{2}\right)}{2}$

iii) $R\ \text{Cos}\frac{A}{2} = \frac{R\left(\text{Sin}\ \frac{A}{2} + \text{Cos}\frac{A}{2}\right) - R\left(\text{Sin}\frac{A}{2} - \text{Cos}\frac{A}{2}\right)}{2}$

iv) $\sqrt{\frac{(R\ \text{Sin}\ A)^2 + (R - R\ \text{Cos}\ A)^2}{2}} = R\ \text{Sin}\ \frac{A}{2}$

v) $\sqrt{\frac{(R\ \text{Cos}\ A)^2 + (R + R\ \text{Cos}\ A)^2}{2}} = R\ \text{Cos}\ \frac{A}{2}$

vi) $\sqrt{\frac{R^2 \mp R.R\ \text{Cos}\ A}{2}} = R\ \text{Sin}\ \frac{A}{2},\ R\ \text{Cos}\ \frac{A}{2}$

Bhāskara II gave no proof to these theorems. The commentator of Kamalākara (1658 A. D.) mentions that Kamalākara gave two proofs of these relations,

1. Sin 2 A = 2 Sin A Cos A (ch. 3, v. 73); Sin 3 A = 3 Sin A − 4 Sin^3A (ch. 3, v. 74); Sin (A ± B) = Sin A Cos B ± Cos A Sin B (ch. 3, v. 68); Cos (A ± B) = Cos A Cos B ∓ Sin A Sin B (ch. 3, v. 69); etc. [in modern equivalents·]

none of which are available.[1] It is also believed that even earlier workers gave similar proofs, but these could not be traced. As corollaries to the general theorem of *samāsa bhāvanā* and *antara bhāvnā*, Bhāskara II indicated how to derive the functions of multiple angles. That Bhāskara II was aware of the cosine formulae has also been, attested by Muniśvara and Kamalākara[2].

Proofs of the above formulae had been derived perhaps as follows :

(1) The proof follows directly if the Radius is taken as the hypotenuse, the *Jyā* as the perpendicular and *Kojyā* as the base of the right angled triangle.

(2) The results follow directly from the definition of *Jyā*, *Kojyā* and *Utkramajyā* of an arc.

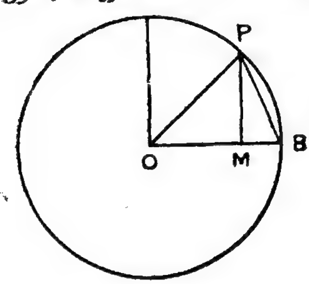

Fig. 20

(3) i) In the right-angled triangle (Fig. 20) PMB,

$PM^2 + MB^2 = PB^2$

or $\frac{1}{2}\sqrt{(PM^2 + MB^2)} = \frac{1}{2}$ PB

or $\frac{1}{2}\sqrt{Jy\bar{a}^2\, A + Utkramajy\bar{a}^2\, A}$

$= Jy\bar{a}\,\frac{A}{2}$ (arc BP = A)

ii) Let K = *Koṭijyā*, U = *Utkramajyā*, R = Radius and J = *Jyā*.

$$\text{Now } K = R - U$$

$$K^2 = R^2 - 2RU + U^2$$

$$\text{or } \quad R^2 - K^2 = 2RU - U^2$$

1. *Siddhāntatattvaviveka* of Kamalākara, ch. 3, v. 70 (Commentary.)

2. *Golādhyāya*, Anandasrama Sanskrit Series No. 122, p. 152; *Siddhāntatattva-viveka*, ed by Sudhakara, p. 113.

or $J^2 = 2RU - U^2$

or $\frac{J^2 + U^2}{4} = \frac{RU}{2}$

$$\sqrt{\frac{J + U^2}{4}} = \sqrt{\frac{RU}{2}}$$

or $$\sqrt{\frac{\text{Jyā}^2\ A + \text{Utkramajyā}^2\ A}{4}} = \sqrt{\frac{R \times \text{Utkramajyā}\ A}{2}}$$

or $$\textit{Jyā}\ \frac{A}{2} = \sqrt{\frac{R \times \text{Utkramajyā}\ A}{2}} \quad \ldots \ldots \text{ by 3 (i)}$$

In modern notation, it becomes,

$$\text{Sin}\ \frac{A}{2} = \sqrt{\frac{\text{Versin}\ A}{2}}$$

(4) Let the two diameters BOB′ and COC′ of the circle of centre O cut each other at right angle (Fig. 21) Let P be any point on the quadrant COB′ such that ∠COP = A. Hence ∠BOP = $90^0 + A$. Let S be the middle point of PB then SB is the *Jyā* of $\frac{90^0 + A}{2}$

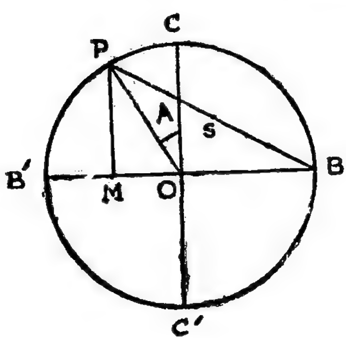

Fig. 21

Now $PB^2 = PM^2 + MB^2$

$= K^2 + (R + J)^2$

where R = Radius,

J = *Jyā* and

K = *Koṭijyā*

of the arc PC.

or $PB^2 = K^2 + R^2 + 2RJ + J^2$

$= R^2 - J^2 + R^2 + 2RJ + J^2$

$$= 2R^2 + 2RJ$$
$$= 2(R^2 + RJ)$$

Again $SB = \frac{1}{2} PB = \sqrt{\frac{2(R^2 + RJ)}{2}} = \sqrt{\frac{R^2 + RJ}{2}}$

i.e., $Jy\bar{a}\ \frac{(90^0 + A)}{2} = \sqrt{\frac{R^2 + R \times Jy\bar{a}\ A}{2}}$

Similarly the other relation my be deduced.

(5) In the Fig. 22, let arc $BQ = A$ and arc $BP = B$. $OP = OQ = R =$ Radius.

Then $QK = QN - KN = QN - PM$

$= Jy\bar{a}\ A - Jy\bar{a}\ B$

$PK = PS - KS = OM - ON$

$= Kojy\bar{a}\ B - Kojy\bar{a} A.$

Now $QP = Jy\bar{a}$ of the arc PQ i.e., A–B

or $\frac{QP}{2} = Jy\bar{a}$ of the arc $\frac{(A - B)}{2}$

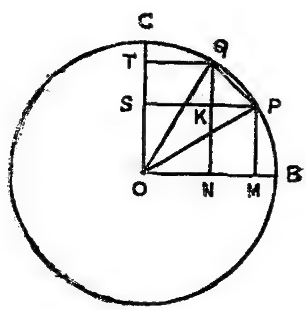

Fig. 22

Since, $\frac{1}{2}\sqrt{QK^2 + PK^2} = \frac{QP}{2}$

we can write,

$$\frac{1}{2}\sqrt{(Jy\bar{a}\ A - Jy\bar{a}\ B)^2 + (Kojy\bar{a}\ B - Kojy\bar{a}\ A)^2} = Jy\bar{a}\ \frac{(A - B)}{2}$$

or $$\frac{1}{2}\sqrt{(\text{Sin } A - \text{Sin } B)^2 + (\text{Cos } A - \text{Cos } B)^2} = \text{Sin } \frac{A - B}{2}$$

(6) Substituting $90^0 - A$ for A, and A for B in the relation (5), the result follows

(7) From (3), we have $\sqrt{\frac{R \times \text{Utkramajy\bar{a} } A}{2}} = Jy\bar{a}\frac{A}{2}$

$$\text{or} \quad R \times \textit{Utkramajyā}\ A = 2\textit{Jyā}^2\ \frac{A}{2}$$

$$\text{or} \quad \textit{Utkramajyā}\ 2\ A = \frac{2\text{Jyā}^2\ A}{R} \quad \text{(Substituting 2A for A)}$$

$$\text{or} \quad R - \textit{Utkramajyā}\ 2A = R - \frac{2\text{Jyā}^2\ A}{R}$$

$$\text{or} \quad \textit{Kramajyā}\ 2A = R - \frac{2\text{Jyā}^2\ A}{R}$$

$$\text{or} \quad \textit{Jyā}\ (90^\circ - 2A) = R - \frac{2\text{Jyā}^2\ A}{R}$$

$$\text{or,} \quad R\ \text{Sin}\ (90^\circ - 2\ A) = R - \frac{(R\ \text{Sin}\ A)^2}{R/2}$$

$$\text{or} \quad \text{Sin}\ (90^\circ - 2A) = 1 - 2\ \text{Sin}^2\ A.$$

(8) In the quarter of the circle (Fig. 23), let the arcs GX = GE = EC. The perpendiculars are drawn from the points, G, E, C on OX and OY.

$$CB = CM + MB$$
$$= CM + IL$$

Now CM : CI = EF : EO

$$\therefore CM = \frac{CI \times EF}{EO} = \frac{GP \times EF}{R}$$

Again, IL : EK = IO : EO

$$\therefore IL = \frac{EK \times IO}{EO} = \frac{EK \times GH}{R}$$

$$(\because IO = GH)$$

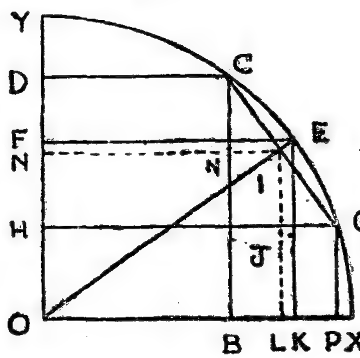

Fig. 23

$$\text{Now } CB = CM + IL = \frac{GP \times EF}{R} + \frac{EK \times GH}{R} \quad \ldots\ldots(1)$$

If the successive *bhujajyās* are denoted by $J_1, J_2, J_3 \cdots$ and successive *Koṭijyās* by C_1, C_2, C_3......respectively,

then $$J_3 = \frac{J_2C_1 + J_1C_2}{R} \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \text{from (1)}$$

By generalising the result,

$$J_{m+n} = \frac{J_m C_n + J_n C_m}{R}$$

or $$Jy\bar{a}\ (A+B) = \frac{Jy\bar{a}\ A \times Kojy\bar{a}\ B + Jya\ B \times Kojya\ A}{R}$$

Mādhava (*Āryabhaṭīyabhāṣya*, TSS 101, p. 58) used the following result to obtain the formula :

$$R\ Sin\ (A \pm B) = \sqrt{(R\ Sin\ A)^2 - (lamba)^2} \pm \sqrt{(R\ Sin\ B)^2 - (lamba)^2}$$ where

$$lamba = \frac{R\ Sin\ A . R\ Sin\ B}{R}$$

Sine Table and its Origin

The construction of the *Jyā* table (or Indian sine table i.e. the values of R Sin θ) or the calculation of the values of the *Jyās* of different arcs in a quadrant was given a great importance by Indian astronomers as it was used to calculate the planetary positions as accurately as possible. For this, the circumference of each quarter of a circle was divided into 24 equal parts, each part covering 225 units (= 3°45′) of the circumference. The twenty-four *Jyā* values as given by Varāhamihira,[1] Āryabhaṭa[2], Govindasvāmī[3] and others for every increase of 3°45′ are arranged in Table I.

1. Bag, A. K. 'A note on the Sine Table I,' *Indian Journal of History of Science*, **7**, No. 1, p. 71–74, 1972; note that in the 5th jyā a term *adhikā* is added.
2. Bag, A. K. 'Sine Table in Ancient India,' *Indian Journal of History of Science*, **4**, Vos. 1 & 2, p. 79–85, 1969; note that 16th jyā according to Āryabhaṭa I and Bhāskara is 2977.
3. Gupta, R. C. 'Fractional Parts of Āryabhaṭa's Sines and Certain Rules found in Govindasvāmī's Bhāṣya on the Mahābhāskarīya', *Indian Journal of History of Science*, **6**, No. 1, p. 51–59, 1971.

TABLE I

The values of R Sin nh (where $h = 3^0\ 45'$, $n = 1, 2, \ldots 24$) as found in different Indian works are tabulated below :

S. No.	*Pañcasiddhāntikā* (505 A. D.) of Varāhamihira	*Sūryasiddhānta* Āryabhaṭa I Āryabhaṭa II Bhāskara II	Brahmagupta (628 A. D.)	Govindasvāmī (850 A. D.)	Mādhava (1400 A.D.) (*Tantrasaṃgraha*) TSS–188 p. 19. ch. 1, V. 15–22
	(1)	(2)	(3)	(4)	(5)
1.	7′51″	225′	214′	224′50″23‴	224′50″22‴
2.	15′40″	449′	427′	448′42″53‴	448′42″58‴
3.	23′25″	671′	638′	670′40″11‴	670′40″16‴
4.	31′ 4″	890′	846′	889′45″ 8‴	889′45″15‴
5.	38′34.5″	1105′	1051′	1105′ 1″30‴	1105′ 1″39‴
6.	45′56″	1315′	1251′	1315′33″56‴	1315′34″ 7‴
7.	53′ 5″	1520′	1446′	1520′28″22‴	1520′28″35‴
8.	60′ 0″	1719′	1635′	1718′52″10‴	1718′52″24‴
9.	66′40″	1910′	1817′	1909′54″19‴	1909′54″35‴
10.	73′ 3″	2093′	1991′	2092′45″46‴	2092′46″ 3‴
11.	79′ 7″	2267′	2156′	2266′38″44‴	2266′39″50‴

TABLE I (Continued)

12.	84'51"	2431'	2312'	2430'50"54"'	2430'51"15"
13.	90'13"	2585'	2459'	2584'37'43"'	2584'38" 6"'
14.	95'12"	2728'	2594'	2727'20"29"'	2727'20"52"'
15.	99'46"	2859'	2719'	2858'22"31"'	2858'22"55"'
16.	103'55"	2978'	2832'	2977'10" 9"'	2977'10"34"'
17.	107'37"	3084'	2933'	3083'12"51"'	3083'13"17"'
18.	110'52"	3177'	3021'	3176' 3"23"'	3176' 3"50"'
19.	113'37"	3256'	3096'	3255'17"54"'	3255'18"22"'
20.	115'55"	3321'	3159'	3320'36" 2"'	3320'36"30"'
21.	117'42"	3372'	3207'	3371'41" 1"'	3371'41"29"'
22.	118'59"	3409'	3242'	3408'19"42"'	3408'20"11"'
23.	119'44"	3431'	3263'	3430'22"42"'	3430'23"11"'
24.	120'	3438'	3270'	3437'44"19"'	3437'44"48"'

In working out the values of *Jyā* table, the circles of different radii (24th jya = jya 90^0 = *Sinus totus*) viz. 120', 3438', 3415', 3437', 44' 19" etc were used by different workers. Since the radius was expressed in minute, the respective *Jyā* length was expressed in minute. The relative accuracy of the values may be judged from Table II in which the values of Sin nh as found from above are compared with their corresponding accurate values.

TABLE II

The values of Sin nh (in decimal)

S. No.	(1)	(2)	(3)	(4)	(5)	Correct values
1.	.06543	.06543	.06544	.06540	.06540	.06540
2.	.13055	.13059	.13058	.13053	.13053	.13053
3.	.19514	.19517	.19511	.19509	.19509	.19509
4.	.25888	.25887	.25872	.25882	.25882	.25882
5.	.32145	.32141	.32141	.32144	.32144	.32144
6.	.38277	.38249	.38257	.38268	.38268	.38268
7.	.44236	.44212	.44220	.44229	.44229	.44228
8.	.50000	.50000	.50000	.50000	.50000	.50000
9.	.55555	.55555	.55565	.55557	.55557	.55558
10.	.60875	.60878	.60887	.60876	.60876	.60876
11.	.65930	.65939	.65929	.65934	.65934	.65934
12.	.70708	.70709	.70703	.70711	.70711	.70711

TABLE II (Continued)

13.	.75181	.75189	.75198	.75184	.75184	.75184
14.	.79333	.79348	.79327	.79335	.79335	.79335
15.	.83139	.83159	.83149	.83146	.83147	.83146
16.	.86597	.86620	.86605	.86603	.86603	.86603
17.	.89681	.89703	.89694	.89687	.89687	.89688
18.	.92388	.92408	.92385	.92388	.92388	.92388
19.	.94681	.94706	.94679	.94693	.94693	.94692
20.	.96597	.96597	.96605	.96593	.96593	.96593
21.	.98083	.98080	.98073	.98079	.98079	.98079
22.	.99153	.99156	.99143	.99144	.99144	.99144
23.	.99778	.99799	.99786	.99785	.99785	.99785
24.	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Varāhamihira gave a fairly accurate (roughly correct upto 4 places of decimals) table of sines though no hint was left how he computed the table. The *Sūryasiddhānta*[1] used the following formula to compute the table :

$$R \text{ Sin } (n+1)h = R \text{ Sin } n\,h + R \text{ Sin } nh - R \text{ Sin } (n-1)h - \frac{R \text{ Sin } nh}{R \text{ Sin } h}$$

1. *Sūryasiddhānta*, ch. 2, v. 15-16 .

where R = 3438′

and n = 0, 1, 2,............23 and R Sin h = h = 3°45′
= 225′

The formulae was parhaps taken up by Āryabhaṭa I. Govindasvamī in his commentary on the *Mahābhāskarīya*[1] of Bhāskara I) and Nīlakaṇṭha in his *Tantrasaṃgraha* (1500 A. D.) added some corrections on the *Jyā* values given by Āryabhaṭa I, which present astonishingly accurate table of Sines (correct upto five places of decimals, vide table II). The Sine table given by other scholars namely, Brahmagupta, Śrīpati, Āryabhaṭa II, Bhāskara II, etc. is no improvement over that of Govindasvamī (c. 800–850 A. D.) How the author of the *Sūryasiddhānta* arrived at the

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1. *saptāgnirandhrāni viyadguṇāgaṃ netrābdhinetraṃ munipañcavedāḥ* ।
dvyakṣyaṣṭasya saṇṇayanadvirāmā vedāgnibhūtam raviṣaṭkṛsānuḥ ।
randhrābhrapakṣaṃ guṇapāvakāṣṭau cakṣurviyatsapta khachandrasūryāḥ ।
rudrāgnicandrā manusaptasoma dasrābhranetraṃ nayanaṃdvi sūryam ।
akṣyabdhipakṣaṃ vasunetrarandhraṃ candrāgnividyā vasukhāṣṭachandram ।
randhreṣuvedaṃ navarūpamidhmaṃ khābhrāgnayas saptaguṇedhmasaṃkhyam ॥
ityuktāstatparādyāḥ syurete hīnādhikāṃśakāḥ ।
guṇānāṃ te tataḥ śodhyā makhyādau yojitā api ॥
tritridvirūpa netrai kadvicandrai kendu saṃkhyayāḥ ।
eka trirūpa netraiśca, jyāvidbhi gaṇakaiḥ kramāt ॥
(*Mahābhāskarīya* of Bhāskara I, ed. by T. S. Kupanna Sastri with the *Bhāṣya* of Govindasvāmī, p. 200–201, Govt. Oriental MS. Library, Madras, 1957).

rule is not very clear. The result of *Sūryasiddhānta* is correct if Sin h of the denominator of the last expression is replaced by 4 $\text{Sin}^2 \frac{h}{2}$.

Trijyā (Sinus Totus)

How the Indian values of R or *Sinus Totus* have been fixed up is not known. Āryabhaṭa I in connection with the verses on instruments made a reference to a circle of radius 57 degrees besides others. Pṛthūdakasvāmī advised to draw a circle of radius 3270 *aṅgulas* with a pair of compass. This appears to be physically impossible if we consider 1 *aṅgula* = 3/4 inches roughly. It may be presumed that the Indian values of R were perhaps obtained from the relation $R = \frac{C}{2\pi}$, where C = circumference = 360 degrees = 21600 minutes.

If we use $\pi = 3.1416$ given by Āryabhaṭa I,

$$R = \frac{360}{2 \times 3.1416} = 57 + \frac{387}{1309} \text{ degrees}$$

$$= 57 \text{ degrees (omitting the fraction)}$$

Though Āryabhaṭa I refers to 57 degrees but no sine table is available where he had used R = 57 degrees.

$$\text{Again, } R = \frac{360 \times 60}{2 \times 3.1416} = \left(3437 + \frac{967}{1309}\right) \text{ minutes} =$$

3438 minutes roughly.

This has been used by Indian scholars for the computation of Sine table.

$$\text{Like wise } R = \frac{360 \times 60 \times 60}{2 \times 3.1416} = \left(206264 + \frac{424}{1309}\right) \text{ seconds}$$

$$= 206264 \text{ seconds approx.} = 3437'44''$$

This has been used by Vaṭeśvara.

$$\text{Again, } R = \frac{360 \times 60 \times 60 \times 60}{2 \times 3.1416} = \left(12375859 + \frac{569}{1309}\right)$$

third $= 12375859 = 3437'44''19'''$

This has been used by Govindasvāmī (c. 800–850).

Mādhava used a far smaller accurate value of π to obtain $R = 3437'44''48'''$. According to *Karaṇa-paddhati* (TSS. 126, ch. 6, v. 7), radius was calculated

$$\text{as } R = \frac{21600 \times 10^{10}}{31{,}41{,}59{,}26{,}536} = 3437'44''48'''$$

Computation of Sine Table :

Several methods for deductions of the rule given in the *Sūryasiddhānta* and *Āryabhaṭīya* have been worked out by Delambre[1], Burgess[2], Singh[3], Naraharayya[4] and Ayyanger[5]. The methods given by last two scholars are reproduced by Srinivasiengar[6]. In a few cases, values given in the *Jyā* (Indian sine) table derived from the rule of the *Sūryasiddhānta* do not strictly agree with the modern values. To remove

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1. *Histoire de l'Astronomie*, **1**, p. 458.
2. Tr. of *Sūryasiddhānta*, p. 62-63, 335, University of Calcutta, 1935.
3. 'Hindu Trigonometry', *Proceedings of the Benares Mathematical Society*, **1**, p. 77–92, 1939.
4. *Journal of Indian Mathematical Society*, **15**, p. 113, second part, 'Notes and Questions', 1924–25.
5. *Journal of Indian Mathematical Society*, **15**, first part, p. 121–6, 1924-25.
6. Srinivasiengar, C. N. *The History of Ancient Indian Mathematics*, p. 51–52, 1967.

the deviation, Srinivasiengar formulated certain subsidiary rules[1].

The values of the sine table based on *Sūryasiddhānta's* rule can be obtained in a much simpler way from the values of *Jyā* $30^0 = \frac{R}{2}$, *Jyā* $45^0 = \sqrt{\frac{R^2}{2}}$, *Jyā* $60^0 = \sqrt{\frac{3}{2}}\ R$ and *Jyā* $90^0 = R$, together with the knowledge of values of two trigonomatrical formulae namely,

Jyā^2 A + *Kojyā*^{2} A = R^2 ... ... (1)

Jyā $\frac{A}{2} = \frac{1}{2}$ (*Jyā*^{2} A + *Utkramajyā*^{2} A)$^{\frac{1}{2}}$ ... (2)

The formulae (1) and (2) and the values for *Jyā* of 30^0, 45^0, 60^0, 90^0 were given by Bhāskara[2] II (1150 A. D.). Somayaji made the observation that it was probably by this method of Bhāskara II that Āryabhaṭa I and his predecessors constructed the sine table. Srinivasiengar pointed out that Somayaji however gave no evidence in support of his observation. Before discussing the problem, it may be pointed out that the calculation is further simplified if the formulae *Jyā* $\frac{A}{2} = \frac{1}{2}$(*Jyā*^{2} A + *Utkramajyā*^{2} A)$^{\frac{1}{2}}$

is replaced by *Jyā*^{2} A = $\frac{1}{2}$(R − *Kojyā*^{2} 2A) ... ... (3)

This may be illustrated by the following example.

Let R = 3438, then *Jyā* $30^0 = \frac{R}{2} = 1719$, *Jyā* $45^0 =$

1. Srinivasiengar, C. N., *Ibid.*, p. 53.

2. Besides these, Bhāskara II gave the correct values of *Jyā* 18^0 and *Jyā* 36^0 as follows :

Jyā $18^0 = \sqrt{\frac{5-1}{4}}$ R and *Jyā* $36^0 = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}}$

(*jyotpatti*, v. 9, 10).

$\frac{R^2}{2} = 2431$, *Jyā* $60^\circ = \frac{\sqrt{3}}{2} R = 2928$, *Jyā* $90^\circ = R = 3438$.

From the *Jyā* values of 45° and 30°, the *Jyā* values of $22^\circ30'$, 15° and then $11^\circ15'$, $7^\circ30'$ and $3^\circ45'$ can be gradually calculated with the help of the formulae (1) and (3). The values of the *Jyā* of the complementary angles vis. $67^\circ30'$, 75°, $78^\circ45'$, $82^\circ30'$, $86^\circ15'$ can be calculated by (2). The *Jyā* of half of these angles and again complements of these angles are calculated. By repeating this process, 24 *Jyās* at the intervals of $3^\circ45'$ can be easily calculated.

It must be noted that the formula (3) readily follows from (2) if we substitute the angle $\frac{A}{2}$ by A. Reverse result follows if we substitute A for 2 A, It is Varāhamihira[1] who first gave the values of *Jyā* 30°, *Jyā* 45°, *Jyā* 60°, *Jyā* 90° and formulae (1) and (3). So it will not be far from truth to state that Varāhamihira (505 A. D.) and possibly other scholars of his time prepared their sine table using these formulae and the values of the *Jyās* mentioned above.

Other methods were also used. Mādhava used the sine and cosine series for the computation of his table to be discussed later.

Ptolemy's Sine Table :

Ptolemy (c. 150 A. D.) first gave a table of chords within a circle of diameter 60 unit and expressed it in sexagesimal unit. The arc ranges from ½ degrees to 180 degrees at intervals of ½ degrees. With the help

1. *Pañcasiddhāntikā*, vide Thibaut's ed. ch. 4, v. 1–5.

of this table, when the length of the arc is known, the corresponding length of the chord can be calculated and vice versa[1]. The chord lengths were given by Ptolemy in terms of its diameter where as in India they were given in terms of its radius. Depending on the similarity of the Greek and Hindu methods, M. Biot[2] opined that this Indian table of *Jyā* was probably obtained from the chords of Ptolemy. Burgess expressed the view "it is rather difficult to calculate the sines given in the *Sūryasiddhānta* from Ptolemy's table of chords. Further Hindu sines differ considerably from that of Ptolemy in few instances"[3]. Burgess is quite justified in his observation the two methods are more or less identical. More light on the subject is likely to clarify further the question under issue. Needham[4] is of the opinion that the Indian work was taken over by the Arabs and by them transmitted to Europe. Since Indian system of place value notation and many elements of the Indian mathematical works were adopted by the Arabs, it is quite plausible that the first impulse of preparing a sine table came to the Arabs from India. Ibn Jābir ibn Sinān al Battāni (c. 858–927 A. D.) used sines and introduced concepts from which the tangent and cotangent can be derived[5].

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1. *Mathematical Syntaxis*, Ch. 11—'A table of lines within a circle' (tr.).
2. *Journal des Savante*, p. 409, 1958.
3. Burgess' tr. of *Sūryasiddhānta*, Calcutta University edition, p. 333.
4. Needham, J. *Science and Civilization in China*, 3, p. 108.
5. Smith, D. E. *History of Mathematics*, 2, p. 608, Dover Publication.

The table of Al-Battani is undoubtedly an improvement over Indian and Greek chords. Abūl-wefa (c. 980 A. D.) computed the table of Ptolemy with much care.[1] By fourteenth century A. D. it was through Peurback (c. 1460 A.D.) and Regiomontanus (c. 1464 A. D.) that the European scholars in general became well acquainted with it.[2]

Method of Interpolation (by Second difference method)

Brahmagupta (628 A. D.) in his *Khaṇḍakhādyaka* (concluding chapter, v. 32) first used the second difference of sines in the method of interpolation.[3] This is undoubtedly an improved step in the line. Brahmagupta's table of sines which runs at an interval of 15° of the arc is as follows :

Arc	*Jyā* Indian Sine	First Difference	Second Difference
0°	0		
15°	39	39	
30°	75	36	—3
45°	106	31	—5
60°	130	24	—7
75°	145	15	—9
90°	150	5	—10

The rule of computing the true tabular difference for calculating the values of the intermediate sines runs as follows :

1. Smith, D. E. *Ibid.*, p. 669.
2. Smith, D. E. *Ibib.*, p. 609.
3. Sengupta, P. C., *BCMS*, 23, p. 125-8, 1931.

gatabhogyakhaṇḍakāntaradagavikalavadhāt
śatairnavabhirāptyā tadyutidalaṃyutonaṃ
bhogyānādhikaṃ bhogyaṃ ।

Translation :

"Multiply half the differencc of the *gatakhaṇḍa* (tabular difference passed over) and *bhogyakhaṇḍa* (tabular difference to be passed over) by the residual arc and divide by 900′ (=h say). The result is to be added to or subtracted from half the sum of the same two tabular differences according as this (semi-sum) is less or greater than the *bhogya-khaṇḍa,* the (final) result is the true functional difference to be passed over".

That is,

$$d = \tfrac{1}{2}(d_t + d_{t+1}) \pm \tfrac{1}{2}(d_t - d_{t+1})\,\frac{\theta}{h}$$

according as $d_t \lessgtr d_{t+1}$ and $f(x+\theta) = f(x) + \frac{\theta}{h}\,.\,d$

combining and putting $\theta = nh$, we get

$$f(x+nh) = f(x) + \frac{n}{2}\Big\{\Delta f(x-h) + \Delta f(x)\Big\}$$

$$+\frac{n^2}{2}\Big\{\Delta f(x) - \Delta f(x-h)\Big\}$$

$$= f(x) + \frac{n}{2}\Big\{\Delta f(x-h) + \Delta f(x)\Big\}$$

$$+\frac{n^2}{2}\Delta^2 f(x-h)$$

The rule employed is equivalent to the Newton-Stirling formula upto second order differences. Munīśvara applied some modified rule to compute his rule.

Illustration :

To find the value of R sin 57°

Now $57° = 3420' = 900' \times 3 + 720'$. Here $\theta = 720'$

$$\text{Now, } d = \frac{31+24}{2} - \frac{720}{900} \times \frac{31-24}{2}$$

$$\therefore R \sin 57° = 106 + \frac{720}{900}\left(\frac{31+24}{2} - \frac{720}{900} \times \frac{31-24}{2}\right)$$
$$= 125.96$$

From logarithmic calculation R sin 57° = 125.80. (here R = 150)

Bhāskara II (1150 A. D.) next quotes Brahmagupta's rule in his *Grahagaṇita* (Spaṣṭādhikāra, v. 16) and takes $h = 10°$ instead of $h = 900' = 15°$ used by Brahmagupta. Brahmagupta has also given a rule for unequal intervals. Govindasvāmī, Mādhava, Parameśvara etc. have given rules of related interest.[1]

Spherical Trigonometry

No problem of spherical trigonometry has been directly discussed by the ancient Indian astronomers. Nevertheless, we find in the *Sūryasiddhānta* and *Āryabhaṭīya* of Āryabhaṭa I certain expressions in connection with the solution of astronomical problems. The values of those expressions could not be deduced without the knowledge of spherical trigonometry.[2] This is demons-

1. Gupta, R. C. "Second order Interpolation in Indian Mathematics upto the fifteenth Century', *Indian Journal of History of Science*, 4 Nos. 1–2, p. 86–98, 1969.
2. Kaye, G. R. 'Hindu Spherical Astronomy', *Journal of the Asiatic Society of Bengal*, 15, new series, p. 153–55, 1919; vide also 'Spherical Astronomy of Āryabhaṭa I' by P. C. Sengupta. *The Cultural Heritage of India*, Ramkrishna Centenary Committee, 3, p. 369–73, Calcutta; Shukla, K. S.

trated by the following formulae relating to sines and cosines of angles of a spherical triangle involving those expressions :

$$\text{i) } \cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\text{ii) } \sin A \sin c = \sin a \sin C$$

$$\text{and iii) } \cos A \sin c = \cos a \cos b - \sin a \cos b \cos C$$

where A, B, C are the angles of a spherical triangle and a, b, c are the corresponding opposite sides. The Indian astronomers employed the sine function principally, and the versed sine occasionally. Sometimes they have used the sine of the complementary angle rather than the cosine function.

Value of π

Historians have devoted much attention to the efforts of the old mathematicians to arrive at approximations to the value of the ratio between the circumference and the diameter of a circle. There is evidence[1] that the ancient Egyptians and old Babylonians had values such as 3.1604 and 3.125, the commonest practice in ancient civilisations was to take the ratio simply as 3. The ancient Indians also used the values varying from $3.1416\left(=\frac{3927}{1250}\right)$, $3.14136\left(=\frac{21600}{6876}\right)$ ....., $3.14285\left(=\frac{22}{7}\right)$........., 3.16227........., 3.0883.........,

'Early Hindu Methods in Sperical Astronomy', *Ganita*, **19**, No. 2, p. 49–72, 1968.

1. Gow, J. *A Short History of Greek Mathematics*, Cambridge, p. 127, 1884; Smith, D. E. *History of Mathematics*, 2, p, 270, Dover Publication; Neugebauer, O. *The Exact Sciences in Antiquity*, Princeton University Press, p. 46, 53, 1952.

3.0885........, and 3. Of these the nearest approximation (3.1416) to the correct value is given by Āryabhaṭa I in the 5th century A. D. None of these values were determined from the ratio of the circumference of the earth to its diameter. On the other hand, they used the π values for determining the earth's diameter from its circumference.[1] Nīlakaṇṭha (1500 A. D.) gave the value $\frac{355}{113}$ in his *Tantrasaṃgraha* (TSS 188, ch. 2, vs 7). Mādhava (1400 A. D.) used the circumference of a circle as 2827433388233 when its diameter was 9×10^{11}, which gives $\pi = 3.14159265359$ correct to 11 places of decimals.

The Greek value of π before Archimedes (c. 250 B. C.) was not satisfactory. Archimedes' values lie between the fraction $\frac{223}{71}\left(=3.1408\ldots\right)$ and $\frac{22}{7}\left(=3.1428\ldots\right)$ which were obtained by the use of 96 sided polygon.[2] Ptolemy[3] (2nd century A. D.) gave an expression for π in sexagesimal unit which is $3^{0}8'30''$. This when expressed in fraction gives out $\frac{377}{120}\left(=3.14166\ldots\right)$ which differs only slightly from Āryabhaṭa I's value of π. It has therefore been

1. Datta, B. 'Hindu Values of π', *Journal of the Asiatic Society of Bengal*, **22**, p. 25–42, 1928.

2. Sarton, G. *Introduction to the History of Science*, **1**, p. 169.

3. Heath, T. L. *History of Mathematics*, **1**, p. 253; Smith, D. E. *History of Mathematics*, 2, p. 308, Dover Publication (Ptolemy accepted also the less correct value).

suggested by many Western scholars like Rodet[1], Heath[2] and others[3] that Āryabhaṭa I's value was borrowed from Greece. G. R. Kaye[4], on the basis of Al-Bīrūnī's statement[5] says that the value given by Āryabhaṭa I occurs in the *Puliśasiddhānta* which was admittedly composed under the Greek influence, supported the above view. The fact that the π values derived from the ratio of the circumference of the earth and its diameter differ substantially from the given values, is also regarded by many of these scholars as a strong argument in favour of the Greek influence on the Āryabhaṭa I's value of π. It should however be pointed out here that the *Puliśasiddhānta* as compiled by Varāhamihira (c. 505 A. D.), the only available text of the said treatise does not make any mention of Āryabhaṭa I's value, though the other values are given.[6] Al-Bīrūnī's statement might therefore be based on a recasted edition of *Puliśasiddhānta* whose authenticity cannot be relied upon in preference to that of the Varāhamihira. Then again as already stated above,

1. Rodet, 'Le'cons de Calcul d'Āryabhaṭa,' *Journal Asiatique*, 7th Series, 13, p. 411, 1878.
2. *History of Greek Mathematics*, **1**, p. 234, 1921 (The word *ayuta* used by Āryabhaṭa I is used earlier than that of the Greek word 'myriad'. It occurs many times in Vedic and Post-vedic literature (vide Macdonell and Keith, *Vedic Index*, **1**, p. 343).
3. Smith, D. E. *History of Mathematics*, **2**, p. 318, Dover Publication.
4. *Journal of the Asiatic Society of Bengal*, **8**, p. 126, 1908.
5. Al-Bīrūnī's *India*—translated by Sachau, **1**, p. 168; **2**, p. 72.
6. *Pañcasiddhāntikā* of Varāhamihira, ch. 4, v. 1.

the Indian workers never calculated their values from the ratio of the circumference of the earth and its diameter. There is therefore no justification for the assumption that Āryabhaṭa I's value was borrowed from Greeks. Needham is also of the same opinion.[1]

As for contemporary development elsewhere, a Chinese scholar Tsu Chhung Chih[2] (430–501 A. D.) gave also the value $\frac{355}{113}$. Whether Indian scholar Nārāyaṇa was influenced by this result is yet to be found.

Later Approximation—After Bhāskara II (c. 1150 A. D.), it is Nīlakaṇṭha who began to realize the irrationality of π. The reason for irrational value of π, according to Nīlakaṇṭha is given as follows: "If the diameter, measured with respect to a particular unit of measurement, is commensurable with respect to that same unit of measurement, the circumference is in-commensurable; and if with respect to any unit the circumference is commensurable, then, with respect to the same unit, the diameter is incommensurable".[3] He therefore proceeded to calculate the value of π in terms of series.

Trigonometrical Series

As the ratio of the circumference and diameter can not be expressed in any commensurable quantity, an

1. Needham, J. *The Science and Civilization in China*, 3, p. 101, see also foot-note.
2. Needham, J. *Ibid.*, 3, p. 101.
3. *yena mānena mīyamāno vyāso niravayavaḥ syāt, tenaiva mīyamānaḥ paridhiḥ punaḥ sāvayava eva syāt yena ca mīyamānaḥ paridhirniravayavastenaiva mīyamāno vyāso'pi sāvayava eva* ।
(*Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, under *Gaṇita*, v. 10)

infinite series for the value of π gradually developed simultaneously with other trigonometrical series namely sine, cosine and tan series. The Sanskrit works, *Tantra-saṃgraha* and *Karaṇapaddhati* gave the expansion of the π, sine, cosine and tan functions, the results of which are almost identical with modern values :

1) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

2) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots$

3) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\ldots$

4) $\theta = \left(\frac{\sin\theta}{\cos\theta}\right) - \frac{1}{3}\left(\frac{\sin\theta}{\cos\theta}\right)^3 + \frac{1}{5}\left(\frac{\sin\theta}{\cos\theta}\right)^5 - \ldots$

where θ is small and $\theta \leqslant 45^\circ$.

The original passages and their translations are given. Proofs of these series have been given in the *Yuktibhāṣā*. The $\frac{\pi}{4}$ series has been discussed in Chapter VI in connection with the development of the concept of integral calculus. Some modern scholars[1] have also made a study of the π series as given in the *Yuktibhāṣā* and expressed the same in modern notation. The tan series known as Gregory's series has been similarly studied also by a number of modern scholars on the

1. Marar, K. and Rajagopal, C. T. 'On the Hindu Quadrature of the Circle', *Journal of the Bombay Branch of the Royal Asiatic Society*, **20**, p. 56–77, 1944; Rajagopal, C. T. 'A neglected chapter of Hindu Mathematics', *Scripta Mathematica*, **15**, nos. 3–4, p. 201-209, 1949.

basis of *Yuktibhāṣā* in which series was worked out with the previous formulation of number of lemmas[1].

π–Series :

This series has been discussed in chapter six of the *Karaṇapaddhati* where the value of π has been expressed in a number of different series. The opening verse runs as follows :

vyāsāccaturghnād bahuśaḥ pṛthaksthāt tripañca-
saptādyayugāhṛtāni ।
vyāse caturghne kramaśastvṛṇaṃ svaṃ kurjāt tadā
syāt paridhiḥ susūkṣmaḥ ॥

(*Karaṇapaddhati*, ch. 6, verse 1)

'Four times of the diameter is to be divided separately by each of the odd integers 3, 5, 7......; every quotient whose order is even, is taken away from the one preceding it. Combined result of all such small operations, when subtracted from four times the diameter, gives the value of the circumference with progressively greater accuracy'.

If C be the circumference and D the diameter, the rule may be expressed as :

$$C = 4D - 4D\left(\frac{1}{3} - \frac{1}{5}\right) - 4\,D\left(\frac{1}{7} - \frac{1}{9}\right) \ldots\ldots$$

$$\text{or } \frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots\ldots\ldots\right)$$

2. Marar, K. and Rajagopal, C. T. *Ibid.*, **20**, p. 56–57, 1949; Rajagopal, C. T. and Aiyar, T. V. V. 'On the Hindu Proof of Gregory's Series, *Scripta Mathematica*, **17**, nos. 1–2, p. 65–74, 1951.

The next verse gives the series in a different form as follows :

vyāsād vanasaṅguṇitāt pṛthagāptaṃ tryādyayugvimūlaghanaiḥ ।
triguṇavyāse svamṛnaṃ kramaśaḥ kṛtvāpi paridhirāneyaḥ ॥

(*Karaṇapaddhati*, ch. 6, verse 2)

'Four times the diameter is divided separately by the cubes of the odd integers from 3 onwards, diminished by these integers themselves. The quotients thus obtained are alternately added to and subtracted from thrice the diameter. The result is the circumference'.

Thus,

$$C = 3D + 4D\left[\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots\right]$$

We can express this also in the following form :

$$\pi = 3 + 4\left[\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots\cdots\cdots\right]$$

$$\text{or, } \frac{\pi-3}{4} = \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \frac{1}{9^3-9} + \cdots\cdots\cdots$$

$$\text{or, } \frac{\pi-3}{4} = \left[\left(\frac{1}{2.3.4} - \frac{1}{4.5.6}\right) + \left(\frac{1}{6.7.8} - \frac{1}{8.9.10}\right) + \cdots\cdots\right]$$

$$\text{or, } \frac{\pi-3}{6} = \frac{2}{3}\left[\left(\frac{1}{2.3.4} - \frac{1}{4.5.6}\right) + \left(\frac{1}{6.7.8} - \frac{1}{8.9.10}\right) + \cdots\right]$$

$$= \left[\frac{1}{(2.2^2-1)^2-2^2} + \frac{1}{(2.4^2-1)^2-4^2} + \frac{1}{(2.6^2-1)^2-6^2} + \cdots\cdots\cdots\right]$$

The last named form of the second series has in fact been given in separate verse as follows:

vargairyujāṃ vā dviguṇairnirekairvargākṛtairvarjita-
yugmavargaiḥ ।
vyāsaṃ ca ṣaḍghnaṃ vibhajet phalaṃ svaṃ vyāse
triṃghne paridhistadā syāt ॥
(*Karaṇapaddhati*, ch. 6, verse 4)

'Six times the diameter is divided separately by the square of twice the squares of even integers (2, 4, 6...) minus one, diminished by the squares of the even integers themselves. The sum of the resulting quotients increased by thrice the diameter is the circumference'.

Thus,

$$C = 3D + 6D\left[\frac{1}{(2.2^2-1)^2-2^2}+\frac{1}{(2.4^2-1)^2-4^2}+\frac{1}{(2.6^2-1)^2-6^2}+\ldots\ldots\right]$$

Convergency of π–Series :

Tantrasaṃgraha adopts various approximations to $\frac{\pi}{4}$. Among these let us discuss the following two which have already been noted by Whish[1]. The two approximations to the series $\frac{\pi}{4}$ are as follows :

$$\text{i) } \frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\ldots\pm\frac{1}{n} \mp \frac{\frac{n+1}{2}}{(n+1)^2+1} \ldots\ldots[T]$$

1. *Transaction of the Royal Asiatic Society* of Great Britain and Ireland, 8, p. 512, 1935.

and ii) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots \pm \frac{1}{n} \mp \dfrac{\frac{(n+1)^2}{4}+1}{[(n+1)^2+4+1]\left(\frac{n+1}{2}\right)}$ ......[T]

In the original series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ (1)

let $S\left(\frac{n+1}{2}\right)$ and $S\left(\frac{n-1}{2}\right)$ denote sums of the first $2m+1$ and $2m$ terms respectively, where $n = 4m+1$. These provide rational approximations to $\frac{\pi}{4}$ where n is large. Let us try to improve them by adding corrections—$f(n+1)$ and $f(n-1)$ respectively, where $f(x)$ is a function to be suitably chosen.

The improved approximations being,

$$T\left(\frac{n+1}{2}\right) = S\left(\frac{n+1}{2}\right) - f(n+1) \qquad (2)$$

$$T\left(\frac{n-1}{2}\right) = S\left(\frac{n-1}{2}\right) + f(n-1) \qquad (3)$$

$T\left(\frac{n+1}{2}\right)$ will be the partial sum of a certain series where general term U_n is given by $T\left(\frac{n+1}{2}\right) - T\left(\frac{n-1}{2}\right)$.

Subtracting (3) from (2), we have,

$$U_n = \frac{1}{n} - f(n+1) - f(n-1) \qquad (4)$$

Changing successively n to $n-2$, $n-4$, ...3, multiplying alternately by 1 and -1, and adding, we obtain,

$$-U_3+U_5-\dots\dots+U_n$$

$$=-\frac{1}{3}+\frac{1}{5}\dots\dots+\frac{1}{n}+f(2)-f(n+1)$$

If $f(n)\to 0$ as $n\to\infty$, we have from (1)

$$\frac{\pi}{4}=1-f(2)-U_3+U_5\dots\dots+U_n\dots\dots \quad (5)$$

Suppose f (x) is assumed to be of the form

$$2f(x)=\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}\dots\dots \quad (6)$$

Expanding $f(n+1)$ and $f(n-1)$ by Taylor's Theorem, we have form (4),

$$\frac{1}{n}=U_n+2[f(n)+\frac{f^{ii}(n)}{2!}+\frac{f^{iv}(n)}{4!}+\dots\dots$$

$$+\frac{1}{(2p)!}f^{2p}(n+\theta_1)-f^{2p}(n-\theta_2)\,] \quad (7)$$

where $0<\theta_1<1$, $0<\theta_2<1$

Since $f(n)=0\left(\frac{1}{n}\right)$, $f^{ii}(n)$, $f^{iv}(n)$ etc. are $0\left(\frac{1}{n^3}\right)$, $0\left(\frac{1}{n^5}\right)$ etc. Hence if we propose that $U_n=0\left(\frac{1}{n^{2p}+1}\right)$, we obtain from (7)

$$\frac{1}{n}+0\left(\frac{1}{n^{2p}+1}\right)=2[f(n)+\frac{f^{ii}(n)}{2!}\dots\dots$$

$$+\frac{1}{(2p-2)!}f^{2p-2}(n)\,] \quad (8)$$

Substituting (6) in (8), we can calculate $a_1, a_2\dots\dots a_{2p}$. We verify that $a_2=a_4=\dots=a_{2p}=0$. Any function of the type (6) can be taken as f (x), after calculating a_1, $a_2\dots\dots a_{2p}$ in this way, the further coefficients a_{2p+1} etc., being chosen arbitrarily in a convenient manner. We then use (4) and (5) to get a series for π in which $U_n = 0\left(\frac{1}{n^{2p+1}}\right)$.

Case i) Putting $p=2$ in (8), we obtain from (6) and (8),

$$a_1 = 1,\ a_3 = -1,\ a_2 = a_4 = 0.$$

Now from (6)

$$2f(n) = \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n^5} - \frac{1}{n^7} + \ldots\ldots$$

$$= \frac{n}{n^2+1}$$

$$\text{Hence } f(n+1) = \frac{\frac{n+1}{2}}{(n+1)^2+1}$$

From (2), we get the approximation

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots \pm \frac{1}{n} \mp \frac{\frac{n+1}{2}}{(n+1)^2+1}$$

Case ii) Putting $p=3$, we obtain as before from (6) and (8)

$$a_1 = 1,\ a_3 = -1,\ a_5 = 5,\ a_2 = a_4 = a_6 = 0$$

From (6),

$$2f(n) = \frac{1}{n} - \frac{1}{n^3} + \frac{5}{n^5} - \frac{5^2}{n^7} + \ldots\ldots$$

$$= \frac{n^2+4}{n(n^2+5)}$$

$$\text{Hence } f(n+1) = \frac{\frac{(n+1)^2}{4} + 1}{\left(\frac{n+1}{2}\right)\left[(n+1)^2+4+1\right]}$$

After this rational approximation, the series becomes,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots \pm \frac{1}{n} \mp \frac{\frac{(n+1)^2}{4} + 1}{\left(\frac{n+1}{2}\right)\left[(n+1)^2 + 4 + 1\right]}$$

This shows that the author of *Tantrasaṃgraha* had definitely the knowledge of slowly converging series of which there is no indication in the *Karaṇapaddhati.*

Sine and Cosine Series :

The following passage containing sine and cosine series is available in the *Tantra-saṃgraha*[1] (1501 A. D.) It has left distinct hints that the results contained in the lines were of Mādhava (c. 1400 A. D.). The verses run as follows :

nihatya cāpa vargeṇa cāpam tattatphalāni ca ।
haret samūlayugvargaistrijyāvargahataiḥ kramāt ॥
cāpam phalāni cādhodhonyasyoparyupari tyajet ।
jivāptyai, saṅgraho 'syaiva vidvān-ityādinakṛtaḥ ॥
nihatya cāpavargena rūpam tattatphalāni' ca ।
hared vimulayugvargaistrijyāvargahataiḥ kramāt ॥
kintu vyāsadalenaiva dvighnenādyam vibhājyatām ।
phalānyadhodhaḥ kramaśo nyasyoparyupari tyajet ॥
śarāptyai, saṅgraho 'syaiva stenastrītyādinā kṛtaḥ ।

English Translation : Multiply the arc by the square of itself (multiplication being repeated any number of times) and divide the result by the product of the square of even numbers increased by that number and

1. Vide *Yuktibhāṣā*, Pt. I, edited with notes by Ramavarma (Maru) and Tampuran and A. R. Akhilesvara Iyre, p. 190, Trichure, 1948.

square of the radius (the multiplication being repeated same number of times). The aιc and the results obtained from above are placed one below the other and are subtracted systematically one from its above. These together give the *jīvā* ($r \sin \theta$) collected here as found in the expression beginning with *vidvān* etc. Multiply the unit (i.e. radius) by the square of the arc (multiplication being repeated any number of times) and divide the result by the product of square of even number decreased by that number and square of the radius (multiplication being repeated same number of times). Place the results one below the other and subtract one from its above. These together give the *śara* ($r—r \cos \theta$) collected here as found in the expression beginning with *stena*.

If t_n and t'_n be the *n*-th expression for *jīvā* and *śara*, then for a small are s and radius r.

$$t_n = \frac{s^{2n}.\ s}{(2^2+2)(4^2+4)\ldots[(2n)^2+2n]r^{2n}}\ (n = 1,2,3\ldots).$$

The successive terms t_1, t_2, t_3...are,

$$t_1 = \frac{s^3}{3!r^2},\ t_2 = \frac{s^5}{5!r^4},\ t_3 = \frac{s^7}{7!r^6},\ t_4 = \frac{s^9}{9!r^8}, \ldots\ldots$$

Then according to the rule,

$$\text{jīvā} = (s - t_1) + (t_2 - t_3) + (t_4 - t_5) + \ldots\ldots$$

$$= s - \frac{s^3}{3!r^2} + \frac{s^5}{5!r^4} - \frac{s^7}{7!r^6} + \frac{s^9}{9!r^8} - \frac{s^{11}}{11!r^{10}} + \ldots\ \ldots\ (1)$$

Again,

$$t'_n = \frac{s^{2n}.\ r}{(2^2-2)(4^2-4)\ldots[(2n^2)-2n]r^{2n}}\ (n = 1,2,3,\ldots)$$

The successive tetms t_1', t_2', t_3', ... are :

$$t_1' = \frac{s^2}{2!r},\ t_2 = \frac{s^4}{4!r^3}, \ldots\ t_6' = \frac{s^{12}}{12!r^{11}} \ldots$$

As per rule, *śara* = $(r - t_1') + (t_2' - t_3') + \dots \dots$

$$= r - \frac{s^2}{2!r} + \frac{s^4}{4!r^3} - \frac{s^6}{6!r^5} + \frac{s^8}{8!r^7} - \frac{s^{10}}{10!r^9} + \frac{s^{12}}{12!r^{11}} - \dots \quad (2)$$

when $s = r\,\theta$, the eqns (1) and (2) reduce to

$$\left.\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \dots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \dots \end{aligned}\right\} \quad \dots (3)$$

Fortunately the passages beginning with *vidvān* and *stena* referred to in the above verses have been preserved in both *Āryabhaṭīyabhāṣya* (TSS 101, p. 113; *yuktibhāṣā*, p. 145) of Nīlakaṇṭha (1443-1545) and *Karaṇapaddhati* (ch. 6. vss. 14–15). In the former it has been clearly stated that the values of the first five terms t_5, t_4, t_3, t_2, t_1 of the eqn (1) and of $t_6', t_5', t_4', t_3', t_2'$ and t_1' of eqn (2) were given by Mādhava (*evāha mādhavaḥ*) when s = 5400′ and r = 3437′ 44″ 43‴. The values are : *vidvā* (= 44‴ = t_5), *tunna bala* (= 33″ 6‴ = t_4), *kavīśanicaya* (= 16′5″ 41‴ = t_3), *sarvārthaśīlasthiro* (= 273′57″ 47‴ = t_2), *nirvirdhāṅga - narendrarung* (= 2220′ 39″ 40‴ = t_1) and *stena* (= 6‴ = t_6'), *stripiśuna* (= 5″ 12‴ = t_5') *sugandhinaganud* (= 3′ 9″ 37‴ = t_4'), *bhadrāṅgabhavyāsana* (= 71′ 43″24 = t_3'), *mīnāṅganarasiṃha* (= 872′3″5‴ = t_2'), *unadhanakṛtbhūreva* (= 4241′ 9″0″ = t_1).

These values when substituted in eqn (1) containing terms from t_1 to t_5, *jīvā* comes out to be 3437′44″48‴, the 24th sine value given in the table of Mādhava (here s = 5400′). Similarly if s is replaced gradually by

225′, 450′, 675′...Mādhava's sine table is obtained. Proceeding in a similar way and substituting values in eqn(2), the cosine table is obtained. This evidently shows that Mādhava, followed by the authors of *Tantrasaṃgraha* and *Karaṇapaddhati*, used the eqns (1) and (2) for the computation of the sine and cosine tables.

How Mādhava arrived at the equations (1) and (2) is not yet definitely known. The *Tantrasaṃgraha* (ch. 2, verse 12½) of Nīlakaṇṭha and *Karaṇapaddhati* (ch. 6, verse 19) have both given that for small arc, *jīvā* $= s - \frac{s^3}{3!r^2}$ (approximately).

The *Yuktibhāṣā* has given the complete rational of the eqns (1) and (2). Its author Jyeṣṭhadeva (c. 1500-1600) in an effort to find an expression for the difference between any arc and its sine chord, divided the circumference of the quarter of a circle into n equal divisions and considered the first and second sine differences. He then found the sum of the first n sine differences and cosine differences by considering all sine chords to be equal to corresponding arc and the small unit of the circumference to be equal to one unit, which evidently gives *jīvā* $= s - \frac{s^3}{3!r^2}$ and *śara* $= \frac{s^2r}{2!r^2}$.

Since sine values are not actually equal to its arc length, further correction was applied *ad-infinitum* to each of the terms of the values obtained for *jīvā* and *śara*, which ultimately gives rise to the eqns (1) and (2) It would not be quite unlikely to presume that the rational was first established by Mādhava before Jyeṣṭhadeva could make use of it.

In western mathematics Newton (1642-1727) is often given credit for the expansion of sine and cosine series No. (3). The result was established later algebraically on a solid foundation by De Moivre (1707-38) and Euler (1748). It is clear from the discussion that the Indian scholar Mādhava (1350-1410) used and possibly established the series (1), (2) and (3) of course in finite form before Newton, De Moivre and Euler, and laid the foundation of his sine table.

Tan Series :

The verse from the *Karaṇapaddhati* runs as follows :

vyāsārdhena hatādabhiṣṭaguṇataḥ koṭyāptamādyaṃ
phalaṃ jyāvargeṇa vinighnamādimaphalaṃ
tattatphalaṃ cāharet ।
kṛtyā koṭıguṇasya tatra tu phaleṣvekatripañcādi-
bhirbhakteṣvojayutaistajet samajutiṃ
jīvādhanuśiśaṣyate ॥

'*Jyā* of the arc (*dhanuṣ*) is to be multiplied by the semi-diameter (*vyāsārdha*) and is divided by the *koṭi*. This is the first term (of the tan series). The value (of the first term) when multiplied by the square of the *jyā* and divided by the square of the *koṭi* gives the second term. This process is repeated. The successive terms are divided by the odd integers 1, 3, 5... . Now when the consecutive terms in the series starting from the first term are alternately subtracted and added gives the circumference'.

In modern notation (Fig. 20), it becomes

$$\text{arc BP} = \text{OP}\left[\frac{\text{PM}}{\text{OM}} - \frac{1}{3}\frac{\text{PM}^3}{\text{OM}^3} + \frac{1}{5}\frac{\text{PM}^5}{\text{OM}^5} - \ldots\right]$$

$$\text{or } s = r\left[\frac{r\sin\theta}{r\cos\theta} - \frac{1}{3}\left(\frac{r\sin\theta}{r\cos\theta}\right)^3 + \frac{1}{5}\left(\frac{r\sin\theta}{r\cos\theta}\right)^5 - \ldots\right]$$

Hence, $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots\ldots$ when $s = r\theta$

The formula holds good if $\theta \leqslant 45^\circ$.

The *Karaṇapaddhati* gives no proof of these series. These series also appear in the *Tantrasaṃgraha* (1465-1545 A.D.) without proof to which Whish already drew attention.[1] The proof of all these can be found in the *Yuktibhāṣā*, an exposition of Nīlakaṇṭha's *Tantrasaṃgraha.*

In the general history of mathematics, the π and tan series are associated with the name of Gregory (1638-1675 A. D.).[2]

Sine and Cosine Series (*Proof*)[3]

In the chapter on Calculus (vide chapter VI) under differentiation, we have shown how to obtain the changes in the sine chord (*bhūjajyā*) and the cosine chord (*koṭijyā*) for small differences in the arc. It has been shown there :—

i) *bhūjakhaṇḍa* or sine difference of 1st order

$$= \frac{R\cos x\,dx}{R}$$

1. *Transaction of the Royal Asiatic Society* of Great Britain and Ireland, 3, pt. III, p. 513, 516, 519.
2. *Chamber's Encyclopaedia*, 10, p. 227, 1935, Cajori, F. *A History of Mathematics*, 2nd revised and enlarged edition, p. 206.
3. Vide also 'The Sine and Cosine Power Series in Hindu Mathematics' by Rajagopal, C. T, and Venkaṭaraman, A. *Journal of the Asiatic Society of Bengal*, 3rd Series, 15, p. 1–13, 1949.

ii) *koṭikhaṇḍa* or cosine difference of 1st order

$$= \frac{R \sin x \, dx}{R}$$

Now the difference between any two *bhūjakhaṇḍas*

$$= \text{Sine difference of the second order}$$

$$= koṭikhaṇḍa \times \frac{dx}{R}$$

$$= R \sin x \frac{(dx)^2}{R^2}$$

This is used in the *Yuktibhāṣā*[1] to find an expression for the difference between any arc and its sine chord. Let x (= AP) be any part of the arc (Fig. 24) in the

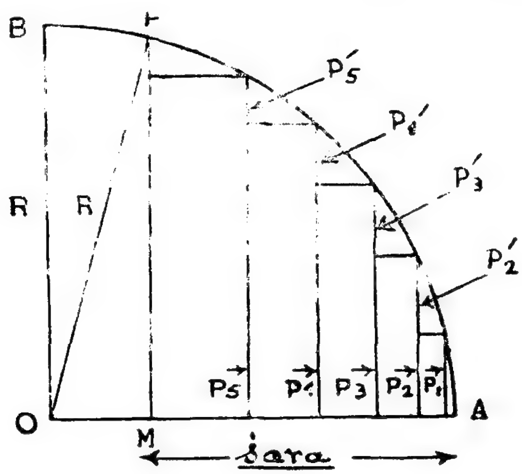

Fig. 24

first quadrant be divided into n equal divisions each being dx. Let $p_1, p_2, p_3 \ldots\ldots p_n$ be the sine chords for parts of the arc x corresponding to parts of the arc

1. Saraswati, T. A. 'Series Mathematics after Bhāskara II', Bulletin of the National Institute of Sciences of India, 21, p. 320-43, 1963.

represented by dx, 2dx, 3dx... . Let $p'_1, p'_2, p'_3 \ldots$ be the corresponding sine differences.

Then $p'_1 = p_1 = R \sin x$

$$p'_1 - p'_2 = R \sin x. \frac{(dx)^2}{R^2} = p_1 \frac{(dx)^2}{R^2}$$

$$\text{or } p'_2 = p'_1 - p_1 \frac{(dx)^2}{R^2}$$

$$= p_1 - p_1 \frac{(dx)^2}{R^2}$$

$$p'_3 = p'_2 - p_2 \frac{(dx)^2}{R^2} = p_1 - p_1 \frac{(dx)^2}{R^2} - p_2 \frac{(dx)^2}{R^2}$$

... ...

... ...

Similarly,

$$p'_n = p_1 - p_1 \frac{(dx)^2}{R^2} - p_2 \frac{(dx)^2}{R^2} - \ldots - p_{n-1} \frac{(dx)^2}{R^2}$$

$$\therefore p'_1 + p'_2 + \ldots p'_n = np_1 - [(n-1)p_1 + (n-2)p_2 + \ldots + 2p_{n-2} + p_{n-1}] \frac{(dx)^2}{R^2}$$

or p_n = sum of all these sine differences

$$= np_1 - [(n-1)p_1 + (n-2)p_2 + \ldots\ldots + 2p_{n-2} + p_{n-1}] \frac{(dx)^2}{R^2}$$

$$= \text{whole arc} - [(n-1)p_1 + (n-2)p_2 + \ldots + p_{n-1}] \frac{(dx)^2}{R^2}$$

If n be sufficiently large, the arc corresponding to p_1 may be approximately equal to p_1, then the approximate length of the whole arc is given by $np_1 = S$ (say).

$$\text{or } S-p_n = \left[(n-1)p_1 + (n-2)p_2 + \dots + p_{n-1}\right]\frac{(dx)^2}{R^2}$$

$$= \left[p_1 + (p_1 + p_2) + (p_1 + p_2 + p_3) + \dots (n-1)\text{times}\right]\frac{(dx)^2}{R^2}$$

or, difference between the arc and sine chord = 2nd order *saṁkalita* of the sines ... ... (1).

Now, *śara* (or *utkramajyā*) of the given arc = the sum of the *koṭikhaṇḍas*

$$= p_1\frac{dx}{R} + p_2\frac{dx}{R} + \dots + p_n\frac{dx}{R}$$

$$= \left(p_1 + p_2 + \dots + p_n\right)\frac{dx}{R}$$

Assuming all the sine chords to be equal to the corresponding arc and dx to be equal to one unit, then the approximate value of *utkramajyā*

$$= \left[1 + 2 + 3 + \dots + (x-1) + x\right]\frac{1}{R}$$

$$= \frac{1}{R}\left[x + (x-1) + \dots + 2 + 1\right]$$

$$= \frac{x^2}{2} \cdot \frac{1}{R} = \frac{x^2}{2R} \qquad \dots \qquad \dots \qquad (2)$$

Expressing sines p'_1, p'_2, p'_3...in terms of the corresponding arcs, we get,

$p'_1 = C_{\frac{1}{2}} \cdot \frac{dx}{R}$ (where $C_{\frac{1}{2}}$ is the cosine chord at the middle point of the small arc dx)

$$= \left(R - h_{\frac{1}{2}}\right) \cdot \frac{dx}{R} \quad \left(h_{\frac{1}{2}} \text{ represents the height of the half chord}\right)$$

Similarly, $p_2' = C_{1\frac{1}{2}} \cdot \frac{dx}{R} = \left(R - h_{1\frac{1}{2}}\right)\frac{dx}{R}$

... ...

... ...

$$p'_n = C_{n-\frac{1}{2}} \cdot \frac{dx}{R} = \left(R - h_{n-\frac{1}{2}}\right)\frac{dx}{R}$$

$$\therefore p'_1 - p'_n = \frac{dx}{R}\left(R - h_{\frac{1}{2}} - R + h_{n-\frac{1}{2}}\right)$$

$$= \frac{dx}{R} \cdot h_n$$

$\left(\because dx \text{ is very small}, h_{\frac{1}{2}} \text{ can be taken as } h_0 = 0 \;\&\; h_{n-\frac{1}{2}} = h_n\right)$

$= \frac{dx}{R}$. *śara* of the given arc.

$$= \frac{dx}{R} \cdot \frac{x^2}{2R} = \frac{x^2 dx}{2R^2} = \frac{x^2}{2R^2} \quad (\text{since } dx = 1 \text{ unit})$$

Similarly, $p'_1 - p'_{n-1} = \frac{(x-1)^2}{2R^2}$

$$p_1' - p'_{n-2} = \frac{(x-2)^2}{2R^2}$$

... ...

... ...

The sum of the left hand side

$$= (n-1)p'_1 - (p'_2 + p'_3 + \ldots + p'_{n-1})$$
$$= np'_1 - (p'_1 + p'_2 + p'_3 + \ldots + p'_{n-1})$$
$$= np_1 - p_n = S - p_n$$

Hence, $S - p_n = \frac{x^2}{2R^2} + \frac{(x-1)^2}{2R^2} + \frac{(x-2)^2}{2R^2} + \ldots$ from (1) or, differences between arc and the sine chord

$$= \frac{1}{2R^2}\left[x^2 + (x-1)^2 + (x-2)^2 + \ldots\right]$$

$$= \frac{1}{2R^2} \cdot \frac{x^3}{3} = \frac{x^3}{3!R^2} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

This gives a closer approximation of the value of the difference between arc x and its sine chord. The difference thus obtained has to be applied as correction to each term of left hand side expression of equation (2) where length of the arc was assumed to be equal to that of its sine chord which was not correct. Hence applying these corrections in equation (2), the expression of the righthand side becomes—

$$\frac{1}{R}\left[\left(x - \frac{x^3}{3!R^2}\right) + \left\{(x-1) - \frac{(x-1)^3}{3!R^2}\right\} + \ldots\right]$$

$$= \frac{1}{R}\left[x + (x-1) + \ldots\right] - \frac{1}{3!R^3}\left[x^3 + (x-1)^3 + \ldots\right]$$

$$= \frac{1}{R}\left[x + (x-1) + \ldots\right] - \frac{1}{3!R^3} \cdot \frac{x^4}{4}$$

$$= \frac{x^2}{2!R} - \frac{x^4}{4!R^3}$$

Hence the correction for the *śara* $= \frac{x^4}{4!R^3}$

When this correction is applied also to the values of sine differences in the equation (3) the expression of the righthand side becomes—

$$\frac{1}{R}\left[\left(\frac{x^2}{2R}-\frac{x^4}{4!R^3}\right)+\left\{\frac{(x-1)^2}{2R}-\frac{(x-1)^4}{4!R^3}\right\}+\cdots\right]$$

$$=\frac{1}{R}\left[\frac{x^2}{2R}+\frac{(x-1)^2}{2R}+\frac{(x-2)^2}{2R}+\cdots\right]$$

$$-\frac{1}{R}\left[\frac{x^4}{4!R^3}+\frac{(x-1)^4}{4!R^3}+\cdots\right]$$

$$=\frac{1}{2R^2}\cdot\frac{x^3}{3}-\frac{1}{4!R^4}\cdot\frac{x^5}{5}$$

$$=\frac{x^3}{3!R^2}-\frac{x^5}{5!R^4}$$

Hence to obtain *jyā* (or the sine chord) the above two corrections are to be applied *ad infinitum* to the independent expressions for the arc sine difference and the *śara.*

Hence

$$jy\bar{a}=x-\frac{x^3}{3!R^2}+\frac{x^5}{5!R^4}-\cdots \qquad (4)$$

And *kojyā* = R − *śara* of the arc

$$=R-\frac{x^2}{2!R}+\frac{x^4}{4!R^3}-\cdots \qquad (5)$$

Now putting $x=R\,\theta$, *jyā* = R sin θ and *kojyā* = R cos θ in (4) and (5), we get respectively,

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots$$

and $$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots$$

Tan Series (Proof) :

$$S = R\ (\tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots)$$

Yuktibhāṣā[1] gives a proof of the series for $\tan\theta = t = 1$. This with a slight and obvious alteration in the figure covering all cases. This alteration is almost the only point of departure from its original proof given below. Proof is based on two lemmas.

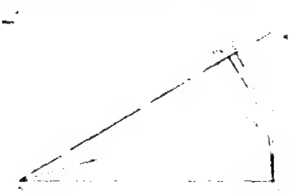

Fig. 25

Lemma 1. Let P,Q be points on the tangent at A to a circle of unit radius, whose centre is 0; let OP meet the circle in p. If m is the foot of the perpendicular from p to OQ, (Fig. 25) pm is given by,

$$pm = \frac{PQ}{OP.OQ}$$

Lemma 2 In Lemma 1, PQ is small, the arc pq of the circle intercepted between OP and OQ is given by,

$$S = \text{arc } pq = \frac{PQ}{OP^2}$$

$$\text{or,} \quad \frac{PQ}{1 + AP^2} (OA = R = 1)$$

Proof : Let $\angle AOB = \theta \leqslant 45^\circ$

or $\tan\theta = AB = t \leqslant 1$

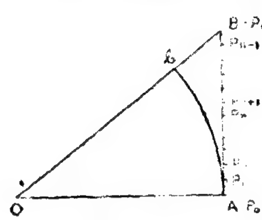

Fig. 26

Divide AB into n equal parts denoting the points of division by $P_0 (= A), P_1, P_2, \ldots\ldots P_{n-1}, P_n (= B)$.

(vide Fig. 26)

Let OB cuts the circle in b.

1. For actual proof vide Mukunda Marar's article, *Teacher's Magazine*, **15**, p. 28-34, 1940, vide also Marar, K. M. and Rajagopal, C. T. "On the Hindu Quadrature of the Circle", *Journal of the Bombay Branch of the Royal Asiatic Society*, **20**, n. s. p. 66–68, 1944.

Then from *Lemma* 2.

$$S = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{P_r P_{r+1}}{1 + AP^2_r}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{\frac{t}{n}}{1+\left(\frac{rt}{n}\right)^2}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t}{n} \left\{1-\left(\frac{rt}{n}\right)^2 + \ldots\right.$$

$$\left. + \left(-1\right)^{\nu-1}\left(\frac{rt}{n}\right)^{2\nu-2} + \frac{\left(-1\right)^{\nu}\left(\frac{rt}{n}\right)^{2\nu}}{1+\left(\frac{rt}{n}\right)^2}\right\}$$

Now making use of the result,

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{r=0}^{n-1} r^p = \frac{1}{p+1}$$

We obtain,

$$S = t - \frac{t^3}{3} + \ldots + (-1)^{\nu-1} \cdot \frac{t^{2\nu-1}}{2\nu-1} + (-1)^{\nu} R_\nu$$

where $0 < R_\nu < \frac{1}{2\nu+1}$

When $n \to \infty$, we get Gregory's series $0 < t \leqslant 1$. The validity of the series for $-1 \leqslant t < 0$ becomes obvious when we change the sign of t.

This testifies to the high degree of development that series mathematics had achieved in India. The development of series mathematics is found in Europe in the works of Newton, Leibniz and Gregory (17th century A. D.) more than a century later. The priority of Indian concept in this connection is therefore beyond question. Further research is necessary to ascertain whether there was any useful contact between India and Europe between fourteenth to seventeenth century A. D. It is not unlikely that such contacts might have occurred during this period when traderoute between the two countries was made through the efforts of navigator like Vasco-da-Gama and others.

CHAPTER VI

INFINITESIMAL CALCULUS

The infinitesimal calculus may be described as a mathematical method of calculating results produced by non-uniform continuous changes as occur in nature. The idea of such method first appeard in the works of Greek scholars like Antiphon (430 B. C.), Eudoxus (370 B. C.), Eudemus (335 B. C) and others. Nearest approach to infinitesimal calculus is first found in the work of Archimedes (225 B. C.). A similar approach to calculus in the middle ages relating to idea of infinitesimal appears in the Indian works, *Siddhānta-śiromaṇi* of Bhāskara II (1150 A. D.) and *Yuktibhāṣā* (c. 1500 A. D.). In Europe in the 17th century A.D. Kepler (1604 A. D.) used the method in a rather primitive form and later in 1629 A.D. Cavaliery made a more clear statement of the idea while describing a method for the calculation of the area of a triangle. In the 17th century A. D., the Japanese scholar Seki Kowa (1642-1708 A. D.) deve'oped a form of calculus known as *Yenri* meaning 'circle principle' to calculate the area of a circle. The subject was finally developed in the modern form by Newton (1664 A. D.) and Leibniz (1676 A. D.).

In India, the idea of 'infinitesimal calculus' arose as Sastri[1], Seal[2] and Sengupta[3] have shown in the attempts of early astronomers to find :

(i) the instantaneous motion of a planet,
(ii) the 'position-angle' of the ecliptic with any secondary to the equator,
(iii) the surface and volume of a sphere, and
(iv) the value of π.

The concept of differentiation appears in the *Siddhāntaśiromaṇi* of Bhāskara II and it was studied in connection with the problem set in items (i) and (ii). The ideas of integration and its development are found in the *Siddhāntaśiromaṇi* of Bhāskara II, *Gaṇitakaumudī* of Nārāyaṇa and the *Yuktibhāṣā* in connection with the problems set in (iii) and (iv).

Differential Calculus :

(a) *Modern concept*—The differential calculus is directly concerned with the momentary state of a phenomenon. This momentary state is symbolised by the differential coefficient. In order to render different processes such as change of position, change of motion, of temperature, volume etc. susceptible to mathematical treatment, the magnitude are supposed to change during

1. Śāstrī, Bapudeva, 'Bhāskara's Knowledge of the Differential Calculus', *Journal of the Asiatic Society of Bengal*, **27**, p. 213-16, 1858,
2. Seal, B. N., *Positive Sciences of the Ancient Hindus*, p. 77-80, Motilal Banarasi Dass.
3. Sengupta, P. C., 'Infinitesimal Calculus in Indian Mathematics', *Journal of the Department of Letters*, Calcutta University, **22**, p. 1-17, 1932.

a series of very short intervals of time. The shorter the interval, the more uniform is the process. The conception is of fundamental importance. The velocity of a train, sixty miles per hour, does not mean that the train can run one mile per minute. The speed of the train might vary in successive minutes, the sum total of which for sixty successive minutes or one hour amount to sixty miles. Hence, the average velocity of the train is one mile per minute. If we continue to reduce the unit of time, the average velocity approaches more and more nearly to the actual velocity of the train during the whole time under consideration. If x_1 be the distance traversed at the time t_1 when the observation commences and x_2 be the distance traversed at the time t_2 then the velocity of the train, $v = \frac{x_2 - x_1}{t_2 - t_1}$. If $t_2 - t_1$ be extremely small then $\frac{x_2 - x_1}{t_2 - t_1}$ may be represented as $\frac{\delta x}{\delta t}$ where δx denotes the space traversed in the small unit of time δt with uniform motion.

Archimedes was the first to give the elementary concept of differential calculus in connection to finding the tangent to a spiral. But no idea of infinitesimal could be found in his method.

(b) *Bhāskara II and his concept of differential*—In the *Spaṣṭādhikāra*[1] chapter of his *Siddhāntaśiromaṇi* Bhāskara II deals with *tātkālikī-gati* (motion for a moment) of a planet which differs very little from the instataneous motion of modern concept of differential calculus.

1. v. 37. see also his own *Vāsanābhāṣya*.

He computed the instantaneous motion of a planet, compared its successive positions and regarded its motion as constant during the time interval between two successive positions. According to Seal[1], this interval cannot be greater than a *truṭi* = $\frac{1}{33,750}$ the part of a second, a unit of time introduced by Bhāskara II himself.[2]

Bhāskara's concept about the motion of the planets will be clear from his own commentary[3] on the verses to calculate the *tātkālikī* motion of a planet. He clearly distinguishes between *sthulagati* (velocity roughly measured) and *sukṣmagati* (velocity measured accurately with reference to indefinitely small quantities of space and time). Here Bhāskara II's *tātkālikī-gati* may be defined as the 'infinitesimal' increment in the planet's longitude due to 'infinitesimal' increment in time.

The following three examples may be quoted from Bhāskara II's work :

(i) Difference between two successive sines given in the table of sines was termed as *bhogya-khaṇḍa* by Bhāskara II. *Bhogya-khaṇḍas* are not equal to each other and the increment of the sine varies with the increment of the arc. Bhāskara II calculated the

1. *Ibid.*, p. 77-78.
2. 1 day = 30 *kṣaṇas*, 1 *kṣaṇa* = 2 *ghaṭikās*, 1 *ghaṭikā* = 30 *kalās*, 1 *kalā* = 30 *kāṣṭhās*, 1 *kāṣṭhā* = 18 *nimeṣas*, 1 *nimeṣa* = 30 *tatparas* and 1 *tatpara* = 100 *truṭis* – (*SiddāntasiromaṅKi*-āla-mānādhyāya).
3. Vide *Siddhāntaśiromaṇi*, Spaṣṭādhikāra chapter, v. 36–37.

values of the *bhogya-khaṇḍas* from the *tātkālika-bhogya-khaṇḍa*[1]. Śāstrī[2] pointed out that *tātkālika-bhogya-khaṇḍa* is the increment in the sine deflected in the direction of the tangent i.e. the instaneous motion of the sine.

To explain this in the quarter of a circle (Fig. 27) let $PP_0 = A$ (or $PP_1 = PP_0$) and arc $XP = y$ and arc $XP_1 = y'$ say, then $P_1S = \sin y' - \sin y = p_1Q_1 - PQ =$ *bhogya-khaṇḍa* of the sine XP and $P_0T = P_0Q_0 - PQ$ = *tātkālika-bhogya-khaṇḍa* (the arc PP_1 is deflected in the direction of the tangent).

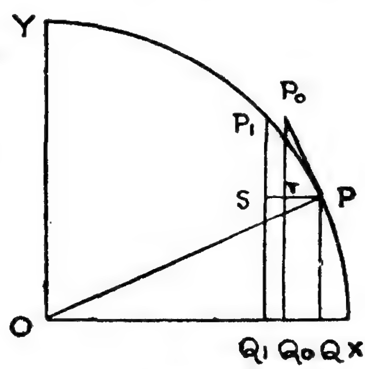

Fig. 27

Now in the two similar triangles POQ and P_0TP

$$= \frac{PP_0 \times OQ}{OP} = \frac{A \times R \cos y}{R} = A \cos y$$

From similar traingles PP_0T and PP_1S

(chord $PP_1 = PP_0$, when the arc is very small)

$$P_1S = \frac{P_0T \times PP_1}{PP_0} = \frac{A \cos y \,.\, (y'-y)}{A} = \cos y \,.\, (y'-y)$$

or, $\sin y' - \sin y = \cos y \,.\, (y'-y)$ which is equivalent to

or, $d(\sin y) = \cos y \, dy$

Changes in the sine chord (*bhujajyā*) and cosine chord (*koṭijyā*) were also calculated by Bhāskara II for small differences in the arc.

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1. *Ibid.*, Spaṣṭādhikāra-Vāsanābhāṣya, v. 36–37.
2. Śāstrī, Bapudeva. *Ibid.*, p. 214.

Let PP_1 be a small increase in the arc PX (Fig. 28) and P_0 be the middle point of the small arc. The prependiculars PQ, P_0Q_0, P_1Q_1 and PS are drawn. From the similar triangles P_1SP and OQ_0P_0, we have

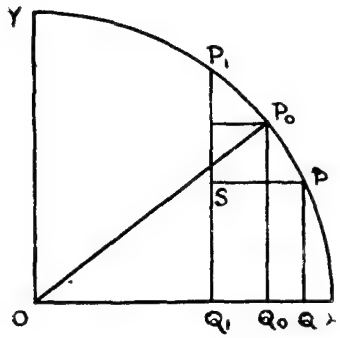

Fig. 28

$$P_1S = \frac{OQ_0 \cdot P_1P}{OP} \text{ and}$$

$$PS = \frac{P_0Q_0 \cdot P_1P}{OP_0}.$$

When the arc PP_1 is very small, the cosine chords at P and P_0 become almost equal and the arc PP_1 coincides with the chord PP_1.

Hence P_1S (*bhūjakhaṇḍa* or sine difference)

$$= \frac{\text{Cosine chord at P} \times \text{increase in the arc}}{\text{radius}}$$

and PS (*koṭijyākhaṇḍa* or cosine difference)

$$= \frac{\text{Sine chord at P} \times \text{increase in the arc}}{\text{radius}}$$

In terms of modern symbols,

$$d(R \sin x) = \frac{R \cos x \ dx}{R}$$

$$\text{and } d(R \cos x) = \frac{R \sin x \ dx}{R}$$

This theorem with proof is also given in both *Āryabhaṭīyabhāṣya*[1] of Nīlakaṇṭha (c. 1500 A. D.) and *Yuktibhāṣā*[2].

1. Vide *Āryabhaṭiya* with Nīlakaṇṭha's commentary under v. 12 of *Gaṇitapāda*.
2. *Ibid.*, p. 165 and 173.

P. C. Sengupta[1] has show that Mūñjāla and his commentator Prasastidhara were aware of the fact that the tabular differences of 'sines' for any given value of the arc are proportional to the cosines. This seems to suggest that Bhāskara II's result in this connection was anticipated by some few centuries ago by Muñjāla, though no clear statement was given by the latter, as to how it was arrived. It might however be pointed out here that even earlier than that, Brahmagupta (620 A. D.) as acknowledged by Bhāskara II himself calculated the average velocity of the planet Mars based on instantaneous motion.

(ii) Śāstrī has given a clear exposition of the *tātkālikī* motion of a planet as conceived by Bhāskara II as follows :

Suppose x, x' = the mean longitudes of a planet on two successive days.

y, y' = the mean anomalies.

u, u' = true longitudes.

a = the sine of the greatest equation of the orbit or ecentricity.

Then $x' - x$ = the mean motion (*madhyagati*) of the planet, $y' - y$ = the motion of the mean anomaly (*manda-kendra*), and $u' - u$ = true motion (*sphuṭagati*) of the planet. Now according to Bhāskara II, the equation of the orbit on the first day $= \frac{a \sin y}{R}$ when R = radius.

1. *koṭiphalaghnī mṛdukendrabhuktistrijyoddhṛtā karki mṛgādikendre* ।
tayā yatonā grahamadhyabhuktistātkālikī mandapari-sphuṭa syāt ॥
(*Siddhāntaśiromaṇi*-Spasṭādhikāra, v. 37)

and that on the next day = $\frac{a \sin y'}{R}$

$$\therefore u = x \pm \frac{a \sin y}{R} \quad \cdots \quad (1)$$

$$\text{and } u' = x' \pm \frac{a \sin y'}{R} \quad \cdots \quad (2)$$

$$u' - u = x' - x \pm \frac{a (\sin y' - \sin y)}{R}$$

$$\text{By (i), } u' - u = x' - x \pm \frac{a \cos y}{R} (y' - y) \quad \cdots \quad (3)$$

This is the instantaneous motion of the planet. Equation (3) is just the differential of equation (1),

$$\text{for } d(u) = d\left(x \pm \frac{a \sin y}{R} \right)$$

$$\text{or, } du = du \pm \frac{a}{R} \cdot \frac{\cos y}{R} \cdot dy,$$

which is equivalent to (3).

(iii) The third case originated in Bhāskara II's

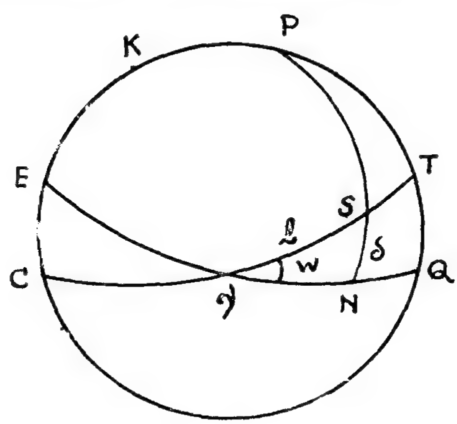

Fig. 29

attempt to find the complement of the angle (Sengupta).

In the Fig. 29

let EQ = Equator, P its pole
CT = Ecliptic, K its pole
and N = any point on EQ.

ν = the first point of aries. It is required to find the complement of the angle νNS In Bhāskara II's terminology, the 'sine' of the required angle is *ayana-valana*. If any point (say S of the ecliptic has a celestial longitude l and δ be its corresponding declination, we have,

$$\frac{\text{Sin } \delta}{\text{Sin } \omega} = \frac{\text{Sin } l}{\text{Sin } 90^0} \text{ i.e. Sin } \delta = \text{Sin } l . \text{ Sin } \omega \quad \ldots \quad (1)$$

Now Bhāskara II's method of finding the *ayana-valana* leads to the differentiation of this equation (1) giving *ayana-valana* to be

$$R\frac{d\delta}{dl} = \frac{R \cos l \ . \ R \sin \omega}{R \cos \delta}$$

(ω = obliquity of the ecliptic)

Leibniz and Newton perfected this concept of differential calculus as we find in modern form.

From what has been discussed above, it follows that Spottiswood has little justification to make the observation that Bhāskara II had no knowledge of calculus.

From consideration of all these facts it may be stated that Bhāskar II's claim as the precursor of Newton and Leibniz in the discovery of the principle of differential calculus as well as in its application to astronomical problems and computations cannot be ignored.

Integral Calculus

Integral calculus is the method of finding the summation of a series when the number of terms in the series tends to infinity, each term being infinitely small.

Although the integral calculus, as we know it to-day arose with Newton and Leibniz in the 17th century A. D., the early stages of its beginning have been traced to the works of Greek scholars namely Antiphon (c. 430 B. C.), Euclid (300 B. C.) and Archimedes (3rd century B. C.)[1]. It is in the works of Archimedes that we find the nearest approach to the concept of actual integration. He calculated the area of a parabolic segment by dividing it into a very large number of small plane geometrical figures and summing up their areas when they are infinitely small[2]. The invention of the concept of calculus was therefore attributed to Archimedes. In India, several centuries after Archimedes, Bhāskara II (1150 A. D) in his *Golādhyāya* has given a similar method of finding the area of a circle, surface area of a sphere and volume of the sphere which involve the ideas of integration[3]. In this method he assumed that the more or less correct result is obtained in these cases by summation of a very large number of small units in the form of triangles for the circle with common vertex at the centre and base on the circumference of the circle and in the form of

1. Heath, T. L. *The works of Archimedes*, pages cxlii – cliv, 235–36, New Dover Edition.
2. Smith, D. E. *History of Mathematics*, 2, p. 679, Dover Publication.
3. Sengupta, P. C. 'Infinitesimal Calculus in Indian Mathematics', *Journal of the Department of Letter*, Calcutta University, 22, p. 1–17, 1932.

pyramids with their vertices meeting at the centre and bases lying on the surface for the volume of a sphere. For calculating the surface-area of a sphere, the method suggested was the summation of the measure of the annular spaces formed by drawing large number of circles of increasing radii around any point on the surface of the sphere covering upto the section passing through the centre of the sphere and multiplying the result by 2. The area of each annular space was calculated by the rule *lambagunaṃ kumukhayogārdhamiti* (i.e., the rule for finding the area of a trapezium). All these obviously involved the idea of summation of the infinitesimal quantities. Hence the results given are in good agreement with the modern values namely,

Area of a circle $= \pi r^2$

Volume of a sphere $= \frac{4}{3}\pi^3$

Surface of a sphere $= 4\pi^2$

But it must be pointed out here that both Archimedes and Bhāskara II though conscious, that the units must be as small as possible for their summation to approach the correct value, they however were not aware of the method of summation of infinite series i.e., the limit of a sum.

The idea of such summation of an infinite series or the limit of the sum of its n terms is, in fact, first found in the *Yuktibhāṣā*[1] (16th century A. D.), nearly a century earlier than Newton. For calculating the value of π, the author of *Yuktibhāṣā* considered a square circumscribing a quarter of a circle of radius x. The

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1. Vide p. 85–99 and p. 181–88.

square was divided by its diagonal. The circumference of the circle included between the diagonal and a side of the square passing through the centre of the circle is $\frac{1}{8}$th of the whole circumference. The side of the square covering this portion of circumference was again divided into large number of very small equal parts by lines drawn from the centre of the circle. Let Δx be the length of each small division on the side of the square which is equal to radius and let $n.\Delta x = x$. In the attempt to calculate the value of :

$\pi\left(\text{since } \pi = \frac{\text{circumference}}{\text{diameter}}\right)$, the author of *Yuktibhāṣā* calculated the value of $\frac{1}{8}$th of the circumference by summing up the large number of small chords corresponding to each small equal division of the circle by applying the properties of similar triangles. The expression for the sum of the chords as deduced in *Yuktibhāṣā* is given by :

$$\begin{aligned} n.\Delta x &- \frac{\Delta x}{x^2}\left[(\Delta x)^2 + (2\Delta x)^2 + (3\Delta x)^2 + \ldots\ldots\right] \\ &+ \frac{\Delta x}{x^4}\left[(\Delta x)^4 + (2\Delta x)^4 + (3\Delta x)^4 + \ldots\ldots\right] \\ &- \frac{\Delta x}{x^6}\left[(\Delta x)^6 + (2\Delta x)^6 + (3\Delta x)^6 + \ldots\ldots\right] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The results of the summation of the infinite number of infinite series within the third brackets as well as the sum of the original infinite series was given in the *Yuktibhāṣā* as follows :

$$\Delta x + 2\Delta x + 3\Delta x + \dots\dots\dots\dots\dots\dots = \frac{x^2}{2}$$

$$(\Delta x)^2 + (2\Delta x)^2 + (3\Delta x)^2 + \dots\dots\dots = \frac{x^3}{3}$$

$$(\Delta x)^3 + (2\Delta x)^3 + (3\Delta x)^3 + \dots\dots\dots = \frac{x^4}{4}$$

and so on.

This summation of infinite series represents the limit of the summation in our modern concept of integral calculus, as for example; $\frac{x^2}{2} = \int_1^{\infty} x\,dx$, $\frac{x^3}{3} = \int_1^{\infty} x^2 dx \dots$ The method of summation of these series has also been given in the *Yuktibhāṣā*.

CHAPTER VII

TREND OF INDO-PERSIAN/ARABIC LITERATURE ON MATHEMATICS IN THE MEDIEVAL PERIOD.

Vast amount of mathematical literature of the medieval period were written in Persian and Arabic and are now available in different oriental libraries in India. Many standard works were also brought in the Sultanate and Mughal Periods from outside world. Attempt was made during this period to prepare translations and write commentaries on these texts. Sultans and Mughal rulers built up hundreds of *maktabs* for elementary education and *madrasahs* for higher education. There were also institutions which laid special emphasis on special field of study like mathematics, astronomy or medicine. Lahore, Delhi and Lucknow became important centres of learning in the period. Education and activities also grew in Jaunpur, Kashmir, Gujrat, Malwa, Bengal, Bihar, Khandwah, and Deccan. In different *madrasahs* subjects like grammer rhetoric, logic, theology, metaphysics, literature, jurisprudence were mainly taught. The teaching of astronomy, mathematics and medicines were not very satisfactory in these *madrasahs*, and Muslims preferred to attend Hindu institutions to master these subjects[1].

1. Jaggi, O. P. *Science and Technology in Medieval India.* Atma Ram and Sons, Delhi, 1977.

Akbar made mathematics compulsory in *madrasahs* for students and issued a *farman*. The study of astronomy and astrology were also encouraged and formed the part of regular courses of study. Inspite of the patronisation, no real effort was taken either to assimilate the knowledge available in Sanskrit sources or from original Persian and Arabic sources were mostly written for people who had no access in standard Sanskrit, Arabic and Persian treatises in mathematics. To what extent the knowledge was available to Indians through Persian and Arabic commentries is yet to be assessed. The following will give an idea of the mathematical activities in the period.

Fīrūz Shah (1351–1388 A. D.) of the Bahmani dynasty was a great patron of astronomical studies. He tried to construct an observatory with the help of Hakim Ali Hasan but it was left unfinished because of Hasan's untimely death.

'Abd al–'Aziz ibn Shams[2] prepared a Persian versian *Tarjuma-hi Bārāhī* of the *Bṛhat Saṃhitā* of Varāhamihira for Fīrūz Shah. Out of the 104 chapters (*bābs*) in the original. only eight were left out in the translation. He might have written another persian work, *Nujūm al–Hind wa San'at-i–usṭurlāb* on Indian astronomy and construction of astrolabe. The date of compilation is not available.

2. Habibullah, A. B. M. 'Medieval Indo-Persian Literature relating to Hindu Science and Philosophy.' *Indian Historical Quarterly*, **14**, pp. 167–181, 1938.

Bābur knew about Ulug Beg's observatory which was established at Samarqand in 1420 A. D. A descrip tion of the observatory is given by Bābur in his Memoir, *Bāburnāmā.* Humāyūn had an extraordinary liking for astronomy. He learnt astronomy and mathematic- under Turkhan Sufidini, well-known scholar in astronomy and mathematics. He discussed astronomy and mathematics with Sufinini, and could handle problems of mathematics and particularly of astrolabe.

Muṣliḥu'd-dīn al-Lārī al-Anṣārī flourished during the reign of Humāyūn (1530–40). Later he joined the court of Mirzā Shāh Ḥusain Arghun (1556), the ruler of Sind He wrote a commentary on Alā'ud-dīn 'Alī Qūshji's *Risālah dar Hai'at* entitled *Sharḥ Risālah i Qūshji.* The manuscript is available in the Bankipur Library and State Central Library, Hyderabad.

The *Lilāvatī* gained wide popularity in India and was held in esteem in the time of Akbar (1556-1605). It is under his order, Abul Faizī, his court poet, prepared a Persion translation *Tarjamah-i-Līlāvatī* in 1587 A. D. (A. H. 996–6). This is mentioned in *Āin-i-Akbari.*

Mullāchānd was also the court astronomer of Akbar. He wrote a work on astronomical table, *Tashīlāt.* This is referred by Farīd-ud-Dīn in *Zīj-i-Shāhjahānī* and Jai Singh Sawāi in *Zij-i-Muḥammad Shāhī.* Nīlakaṇṭha Jyotirvid who flourished in the court of Akbar compiled a *Tajik* work in which a large number of Persian technical terms were introduced.

'Atāu'llah Qārī (Qadiri ?) flourished during the reign of Barhān Niṣām Shāh, ruler of Aḥmadnagar

(1591–1595). His *Risāla dar Ma'rifat-i A'māl-i Rub'mu-jayyā-i Āfāqī* deal with sine quadrant. The manuscript is available in the State Central Library, Hyderabad and Azad Library, Aligarh.

A Persian translation of the *Bījagaṇita* entitled *Tarjamah-i-Bīj* was made in India in 1634-35 A. D. by Atāullāh Rashīdī, son of Ustad Aḥmad Nādir, the builder of Taj Mahal He was originally a resident of Lahore and had his education under his father, and Makramat Khan, who was well-known for his knowledge in mathematics. The work was dedicated to Shāhjahān (1628–1659). The Persian version was translated into English by Edward Strachey in 1813 A. D. From the traslation it appears that it is not a faithful translation of the *Bījagaṇita* but is a mixture of text, commentary and some interpolation. The same author wrote perhaps two other works viz. *Khasinatril A'dad* dealing with arithmetic, algebra, and applied geometry. Some discussion is also available on some astronomical problems. The other work is *Khulasah-i-Baz* dealing with arithemetic, mensuration and algebra in verse form. The manuscript copies are available in Bankipur and British Museum.

Farīd'd-dīn Ma'sud, son of Ḥāfis Ibrāhim Munajim was the court astronomer of Shāhjahān. He was considered as a great scholar in mathematics, astronomy, geometry. etc. He wrote perhaps two works viz. *Sirājul Istikhrai and Zij-i Shāhjahānī* in 1629 A. D. The former deals with eras, calculations of dates and principles of computation of almanacs. The latter is on calendars and astronomical tables. The manuscripts of both the works are available in the State Central Library, Hyderabad.

Lutfu'llah Muhandis, the brother of Atā'ullāh Rashīdī (c. 1634) and the second son of Ustad Aḥmad, was also well-known for his interest in mathematics and other sciences. His *Muntakhab* is a short versified translation of Bahā'ud-dīn Āmulī's mathematical work *Khulāṣatu'l Ḥisāb* and was written in Persian in 1681 A. D. The copies of the manuscripts are available in the Asiatic Society of Bengal, Bankipur, British Museum, Azad Library (Aligarth) and Salarjang Library (Hyderabad). His *Risāla dar Jawab i-Sawali* on geometry in question answer form and *Risāla-i-Arsmatiqi* on properties of numbers are still available in the Rampur Library and Saidiyah Library (Hyderabed). He wrote *Taqwim Luṭifi,* on almanac and *Tarjuma Kitāb Suwar-i-Kawākib,* a commentary on the celebrated astronomical work *Suwaru'l Kawākib.* The manuscript is available in Rampur Rida Library (U. P.) and Azad Library (Aligarh).

Hāji Kha'ullah, son of Aman'ullah and brother of Mulla Murshid Makramat Khan in the beginning of 17th century A. D. (died in 1649) wrote a commentary on the work of mathematics is entitled *Sharḥ-i-Kitāb-i-Haji Kalil.* A copy is available in the Rampur Library.

Sh. Muḥammad (17th century A. D.), son of Sh. Muḥammad Saīd flourished under emperor Aurangzeb (1659–1709) He wrote in Arabic *Shārḥ as Sirājīya.* an incomplete commentary on as-Sajāwandī's algerbaical treatise known as *Sirājīya.* The manuscript is available in the Asiatic Society of Bengal.

Dharma Nārāyan ibn Kalyānmal Kayath wrote a Persian commentary in 1663–64 A.D. at Etawah on the *Līlāvatī* under the title *Badā' i-i-Funūn* and dedicated to Alamgir Aurangzeb (1659–1709). The copies of manuscripts are listed by C. A. Story. This shows that the *Līlāvatī* received recognition among the Mughal emperors. Manuscript copies of Faizī's version are deposited in the British Museum, India Office Library and John Rylands Library in Manchester to mention a few. Another version *Dastūr al Ḥisāb : Tarjuma-i-Līlāvatī* was prepared by Amīn b Shaikh Muḥammed Sa'īd in 1678 A. D. The incomplete Manchesther copy has been translated by Winter and Mirza, the work contains a selection of examples taken from the *Līlāvatī.* The examples include problems on investigation of mixture, rule of three, inverse proportion, compound proportion etc. and concern primarily business transaction.

Khawāja Bahādur Husain Khān Bahādur flourished during the reign of Aurangzeb and later went to Qulīch Khān. He learnt astronomy and astrology under S. Alavī Khān Zubadatul Munajjimin Ṭāliqānī. He is known for his two works viz. *Sharḥ-i zīj-i Nizāmī* (commentary on *Zīj-i Nizāmī*) and *Zīj-i Nizāmī* (astrological and astronomical table). The scholar has studied of both Indian and non-Indian works. The copies of both the works are available in the State Central Library.

Nand Rām, son of Hiranand Ka'isth flourished during the reign of Mughal emperor Aurgangzeb and wrote a work on accountancy, *Ain-i-Siyaq* in 1680 A. D. One copy of the manuscript is available in the State Central Library, Hyderabad.

M. Husain s|o Khalilullāh (d. 1696) was born at Bījāpur and studied under M. Zubair at Bījāpurī. He was appointed principal of Madrasah-i-Maḥmūd in Bidar by Aurgangzeb in 1686 A. D. He produced *Ujalatur Rub* in Arabic which deals with application of quadrant for recording various astronomical data. The manuscript is available at Sa-Idiyah Library, Hyderabad

Rājā Jayasiṃha (1693-1743) flourished under the patronization of Maḥammad Shāh (1719-1748 A. D.) and tried to rectify and improve the almanacs already constructed by his predecessors.[1] He started organising new observations with the help of the Muslim, Hindu and European experts. After seven years of observations in Delhi, Jaipur, Mathura, Benares, and Ujjain, he deputed Padre Manoel with some competent hands to Europe who brought back with them the astronomical tables of De la Hire. He was also familiar with Euclid's *Elements of Geometry*, Ptolemy's *Almagest*, certain writings on astrolabe, the tables of Ulugh Beg of Samarkand and Flamsteed's *Historia Celestis Britannica*. The materials are compiled in his *Zīj-i Jādīd –i Muḥammad Shāhī*, which was completed in 1727 A. D.

1. Kaye G, R (1) *The Astronomical Observatories of Jai Singh*, Archaeological Survey of India, New Imperial Series No. 40, Calcutta, 1918.
(2) *A Guide to Old Observatories at Delhi, Jaipur, Ujjain & Benares*, Calcutta, 1920.

The manuscript is available at the Oriental Library, Bankipur and British Museum. Under his patronization, Samrāṭ Jagannāth translated Ptolemy's *Syntaxis* as *Siddhāntasāra Kaustubha*, Euclid's *Elements* as *Rekhāgaṇita*, Kewal Ram (Gujrathi Brāhman) translated De la Hire's table as *Jai Vinod Vibhag Sāraṇi*, Ulugbeg's tables as *Tārā Sāraṇi*. Pundarik Ratnākar, a Maharastrian Brahmin wrote *Jai Singh Kalpadrum* dealing with Purāṇic facts.

Muḥammad Zaman Fayyad, son of M. Sadiq al-Anbalaji ad-Dehlawī wrote in 1718 A. D. his *Ghayat-i Juhdu'l Ḥisāb*. The manuscript is available in the Bankipur and Rampur Libraries. He was a native of Ambala and later on resided at Delhi. He wrote several other books in mathematics and astronomy. His *Taḥrīru'l Ashkāl li Hal-li Sharḥ-i Ashkālu't Ta' sis li Ṭusī* is a super-commentary on the commentary by Ṭusī on the *Ashkālu't Ta 'sis*, a geometrical work of *Shamsu'd–Dīn* M. b. Ashraf Husaini. The manuscript is available in the Rampur Library.

'Imāmu'd-dīn Ḥusain (b. 1701), the eldest son of Luṭfu'llāh Muhandis was a well-known astronomer. He wrote many works on astronomy, of which his *at-Tā' liqāt alā Sharhi'l-Mulakkhasi'l Chagmīnī* is a commentary on the *al-Mulakhkhas fi'l Hai'al of Qāḍī Ẓadah ar-hūmī* and *at-Taṣrīh fi sharhi Taṣhrīḥ* is a commentary on *Taṣrīḥu'd-dīn Aflāk* of Bahā'ud-dīn Āmūlī.

Mīrzā Khairullāh Muhandis (c. 1700-1740), the second son of Luṭfalla Muhandis of Lahore and nephew of 'Ata' Allāh Rushdī (c. 1634) was astronomical adviser to Jayasiṃha (1693-1743) and wrote a *Sharḥ* on the latter's *Ẓīj-i-Muḥammad Shāhī*. He also translated a copy of *Almagest* and wrote a commentary on it. A manuscript of *Almagest* with his commentary is available at the Raza Library, Rampur.

Abu'l Khairu'llah son of Luṭfu'llah Muhandis was appointed head of the observatory at Delhi by emperor Mohammad Shāh in 1718. His *Taqrību't Taḥrīr*, is a Persian translation of Naṣīrud-dīn's Arabic version on Ptolemy's *Almagest*. He compiled with the help of Nizāmu'd-dīn at Barjandi's commentary on Ṭusī's above work. The manuscript is available at the Bankipur Library. He wrote another commentary on the *Zīj-i-Muḥammad Shāhī*.

Mulchand, son of Harihar Prasad flourished during the reign of Muḥammad (1719-1748), ruler of Delhi. His *Ḥisāb Nāmāh*, a treatise on arithmetic was written in Delhi. One copy of the manuscript is available in State Central Library.

Anand Ram Mukhlis, son of Rajah Mardi Ram of Allahabad flourished during the reign of Muḥammad Shāh (1719-1748). His work *Dasturu'l Amal* chiefly dealing with accountancy was written in first half of 18th century in Persian. It contains information on weights and measure, zodiacal signs, Hindu science and śāstras in tabulated form. He quoted profusely about his teacher Mirza Bedi in the work.

Inderman, a native of Hisar wrote his *Dastur-i-Ḥisāb* in 1767 A. D. It is a treatise in five *maqalah* and a *khatimah*. One copy is available at Bankipur Library.

Muhammad Barkat flourished in Lahore in 1782 A. D. and was well-known for his *Sharḥ Taḥrīr-u usuli'l Handasat-i wa'l Ḥisāb*, a commentary in Arabic on the first book of *Euclid* and *Al-Hashiyah'ala Uqlidas,* a gloss on the *Euclid.* The manuscripts are available in the Osmania University Library.

Khwajah Muḥammad was a native of Hydarabad and dedicated his arithmetical work, *Mir'atu'l 'Ḥisāb* in 1786 A. D. to Mumtazu'd Daulah M. A. 'Zamu'd-Din Khan Bahādur Muzaffar Jung, commander in chief of Nizāmud Dīn Mir. Nizām 'Alikhan Fath Jung, ruler of Hyderabad. The manuscript is available at the Azad Library, State Central Library, Rampur Library and Mashriqi Kutub Khanah Salar Jung (Hyderabad). He wrote also a commentary *Sharḥ-i-Khulāsatu'l Ḥisāb* on the *Khulāṣatu'l Ḥisāb of Bahā'ud-Dīn 'Āmulī'.* The manuscript is available in the State Central Library.

Raushan Ali (fl. second half of the 18th century) was born at Jaunpur. He taught at Calcutta. Madrasah and Fort William College. He wrote a number of works in mathematics and other subjects. Some of these works are *Risālah fi Jabr wa Muqablah* (on Algebra), *Risāla-i-Ḥisāb* (a treatise on arithmetic) *Tarjumḥ-i Khulāṣatu'l Ḥisāb* (a translation of Bahā'ud-dīn Āmulī's mathematical treatise). The manuscript is available in Rampur Library.

Karim Bakhsh made a selection from a larger treatise, *Umudu'l Ḥisab* for a Deccan Prince Arastu Jah. Three copies of the manuscripts are available in the Sa Idiyah Library, Hyderabad.

Nawab Shamsu'ul Umara Fakhru'd-dīn Khān Bahādur (b. 1785) was a decendant of Faridu'd-dīn Mas'ud al-Ajudhani. His grand father migrated to

Hyderabad and was appointed an officer under Asaf Jah Nizam, first founder Nizam of Hyderabad (died in 1748). He took interest in propagating western knowledge to India. He wrote two works viz. *Risālah der Bayan i Amal-i Qite*, a treatise on the construction of the sector and *Sham su'l Handasah*, a work on geometry mensuration and trigonometry. The copies of both these manuscripts are available in State Central Library, Hyderabad.

M. Ḥusain Isfahani Landani, son of S. Abdu'l-'Azim Isfahani Landani flourished during the days of Asifu'lmulk Sikandar Jāh Bahādur. He wrote his *Risālah-i Haī'at-i Angrezī* in 1797 A. D. on European astronomical system specially English and French. The manuscript is available in State Central Library, Hyderabad and Rampur Library, U. P.

Sayyid Nuru'l Asfiyah (c. 1800 A. D.) was born at Aurangabad. He lived for a considerable time with Nawab ali Khān at Karnal and later on shifted to Hyderabad and joined the service under Nawab Shamesu'ul Umara. He wrote *Risālah-i-Nuru'l Ḥisāb*, a treatise on arithmetic. The manuscript is available in the State Central Library, Hyderabad.

Sh. Aḥmad b. M. Maghribi Tilimsani al-Ansari as -Sa'imi (c. 1814) was attached as collector in the Department of Revenue of Madras. He composed several treatises on mathematics and astronomy. His *A'zamu'l-Ḥisāb* is a treatise on mathematics, now available in the State Central Library. The *Zubdatu'l Ḥisāb* is another mathematical treatise available in Asiatic Society in four chapters dealing with arithmetic, measurements, finding an unknown quantity and some essentials relating to arithmetic. He perhaps wrote another work *Mir'atu'l-'Alam* on mathematics. The manuscript is available in the State Central Library.

Abul Qāsim (Ghulam Ḥussain), son of Fath M. Al-Karbala-i wrote his *Jami'i Bahādur Khānī*[1] in 1834 A. D. He was born at Jaunpur in 1790-91 A. D. and had his lesson in mathematics under his father and some contemporary scholars in mathematics. He spent most of his time with the princes of Benares and Murshidabad The work is devoted to mathematics and astronomy in six chapters viz. science of geometry, optics, arithmetics, practical gemoetry dealing with the measurement and division of circle, heavenly bodies, horoscope and calendar. The work starts with the element of geometry and arithmetic according to Hindus, then European methods of decimal fractions, logarithms, trigonometrical tables and then gives a system of astronomy first according to Hindus, then according to Ptolemy, then according to Copernicus together with an account of astronomical instruments and calculations on astronomical almanacs. The copies of manuscripts are available in the Asiatic Society of Bengal, State Central Library, Salarjung Library, Hyderabad. He wrote several other works viz. *Sharzhala Taḥrīr-i Uqlīdas* (commentaries to *Euclid*) and *al-Mijisṭī* (commentary to *Almagest* of Ptolemy). *Anīsu'l-Aḥbāb*

1. Tytler, John. 'Analysis and Specimens of a Persian Work on Mathematics and Astronomy, *Royal Asiatic Society of Great Britain and Ireland*, 5, pp. 254-72, 1837.

fi Bayān-i Masā'il-i Usṭurlāb (commentary on the Bahāud-dīn Āmulī's treatise on *Sufaiḥah*), *Iṣṭilāḥātu't Taqwīn* (on compilation of almanacs) and *Ẓij-i-Bahādur Khānī* (on astronomical tables).

To sum up, the present chapter gives an incomplete survey of the Persian and Arabic literature on mathematics during the medieval period in India. It is by no means complete. The Persian and Arabic literatures were produced mostly under the patronage of Mughal rulers.[1] Many standard works were brought from outside India. Some of these are *Khulāṣtu'l Ḥisāb* of Bahā'ud-Dīn Āmulī (c. 1547-1621) written originally in Arabic in Iran, *Taḥrīr-i-Uqlidas* and *al-Mijisṭī*, Arabic version of Euclid's *Elements* and Ptolemy's *Almagest* by Nasīru'd-dīn aṭ-ṭūsi etc. besides some others on acountancy, which attracted attention of many Indian scholars.

Attempt has been made to translate and write commentaries on these texts. Similar attempts have been tried to make translations of *Bṛhatsaṃhitā*, *Līlāvatī* and *Bījagaṇita* in the period, but very few attempts have been made to make a comparison with the available knowledge in Sanskrit sources. These were written mainly for readers of Persian who knew no other language and had no access in standard Sanskrit, Arabic and Persian treatises in mathematics. Only a partial attempt has been made by Muniśvara and Kamalākara, Jagannātha Paṇḍita and Rājā Jayasiṃha to make a synthesis of the available Indian knowledge, and that of Ptolemy and Euclid. The account gives an idea of

1. Storey, C. A. *Persian Literature*, Vol. 2, pp. 1, 4–5, London, 1938.

both Hindu and Islamic traditions in India as well as their activities in the form of writing commentaries on older Indian and some Persian texts, which helps us to some extent to assess the trend of literature. The activities of the Indian scholars in mathematics also attracted the attention of European scholars viz. Giovanni Dominique Cassini (1691-1699), Le Gentil (1772), Robert Barker (1777), Joseph Tieffenttaler (1785-1789), Bailey (1787), William Jones (1790), Samuel Devis (1790-1892) and John Bentley (1799), who tried to make assessment of Indian activities by writing translation of the text, writing articles in French, Latin and German languages. Sen[1] has made a resumé of these activities. But the assessment of the actual contribution in the period deserves more intensive research. This is very important because this will help us to assess correctly the proportion of cultural interdependence in the field of mathematical knowledge.

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